CBSE Examination Paper, 2018

General Instructions:

(*i*) All questions are compulsory.

(*ii*) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.

(*iii*) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(*iv*) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.

(*v*) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Note: All the questions in set 2 & set 3 are same. Only the order of questions is different.

1. Find the value of $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$.

Sol.
$$\begin{aligned}\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) &= \tan^{-1}\sqrt{3} - [\pi - \cot^{-1}(\sqrt{3})]\\ &= (\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}) - \pi\\ &= \frac{\pi}{2} - \pi = \frac{-\pi}{2}.\end{aligned}$$

2. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew symmetric, find the values of 'a' and 'b'.

Sol. For skew symmetric matrix, $a_{ij} = -a_{ji} \Rightarrow a = -2, b = 3$.

3. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$.

Sol.
$$\vec{a} \cdot \vec{b} = \frac{9}{2} \Rightarrow |\vec{a}||\vec{b}| \cos 60° = \frac{9}{2}$$

$$\Rightarrow \quad \frac{1}{2}|\vec{a}|^2 = \frac{9}{2} \Rightarrow |\vec{a}| = 3 = |\vec{b}| \qquad (\because |\vec{a}| = |\vec{b}|)$$

4. If $a * b$ denotes the larger of 'a' and 'b' and if $aob = (a * b) + 3$, then write the value of $(5)o(10)$, where $*$ and o are binary operations.

Sol. $$(5)o(10) = (5 * 10) + 3 = 10 + 3 = 13.$$

5. Prove that: $3 \sin^{-1} x = \sin^{-1}(3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Sol. Let $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$$\text{RHS} = \sin^{-1}(3x - 4x^3) = \sin^{-1}[3\sin\theta - 4\sin^3\theta]$$
$$= \sin^{-1}[\sin 3\theta] = 3\theta = 3\sin^{-1}x = \text{LHS}$$

6. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$.

Sol. Given
$$A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$
$$|A| = \begin{vmatrix} 2 & -3 \\ -4 & 7 \end{vmatrix} = 14 - 12 = 2 \neq 0$$

and
$$\text{Adj}\,A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$\therefore$
$$A^{-1} = \frac{1}{|A|}\text{adj}\,A = \frac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \quad ..(i)$$

Also
$$9I - A = 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 9-2 & 0+3 \\ 0+4 & 9-7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii), we get
$$2A^{-1} = 9I - A.$$

7. Differentiate $\tan^{-1}\left(\frac{1 + \cos x}{\sin x}\right)$ with respect to x.

Sol. Consider
$$y = \tan^{-1}\left(\frac{1 + \cos x}{\sin x}\right)$$
$$= \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]$$

$$\Rightarrow \quad y = \frac{\pi}{2}-\frac{x}{2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}.$$

8. The total cost $C(x)$ associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Sol. Given $\quad C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$

For marginal cost, $\quad C'(x) = 0.015x^2 - 0.04x + 30$

$\therefore \quad C'(3) = 0.015 \times 9 - 0.04 \times 3 + 30$

$$= 0.135 - 0.12 + 30 = 30.015.$$

9. Evaluate $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\,dx$

Sol. Consider $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}dx = \int \frac{1 - 2\sin^2 x + 2\sin^2 x}{\cos^2 x}dx \quad (\because \cos 2x = 1 - 2\sin^2 x)$

$$= \int \frac{1}{\cos^2 x}dx$$

$$= \int \sec^2 x\,dx = \tan x + C$$

10. Find the differential equation representing the family of curves $y = ae^{bx+5}$, where a and b are arbitrary constants.

Sol. Consider $\quad y = ae^{bx+5}$

On differentiating both sides, w.r.t, x, we get

$$\frac{dy}{dx} = abe^{bx+5} = by \quad ...(i)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = b \cdot \frac{dy}{dx} \quad ...(ii)$$

From (i) and (ii), eliminating b, we get

$$y \cdot \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \text{ as required equation.}$$

11. If θ is the angle between two vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$, find sin θ.

Sol. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = \hat{i}(4) - \hat{j}(-8) + \hat{k}(4) = 4\hat{i} + 8\hat{j} + 4\hat{k}$$

$\therefore$ $|\vec{a} \times \vec{b}| = 4\sqrt{1+4+1} = 4\sqrt{6}$

Also $|\vec{a}| = \sqrt{1+4+9} = \sqrt{14}$; $|\vec{b}| = \sqrt{9+4+1} = \sqrt{14}$

$\therefore$ $\sin\theta = \dfrac{4\sqrt{6}}{\sqrt{14}\sqrt{14}} = \dfrac{2\sqrt{6}}{7}.$

12. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Sol. A : sum 8, i.e. (2, 6), (3, 5), (4, 4), (5, 3), (6, 2)

B : red die number less than 4.

B : (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3).

$A \cap B$: (6, 2), (5, 3)

$\therefore$ Conditional probability of obtaining sum 8, given that red die resulted in number less than 4.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{18}{36}} = \frac{1}{9}$$

13. Using properties of determinants, prove that

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx)$$

Sol. Consider $\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}$

$$= \begin{vmatrix} 0 & -3x & 1+3x \\ 3y & 0 & 1 \\ -3z & 3z & 1 \end{vmatrix} \quad \text{[by performing } C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3]$$

$$= \begin{vmatrix} 0 & -3x & 1 \\ 3y & 0 & 1 \\ -3z & 3z & 1+3z \end{vmatrix} \quad \text{[by performing } C_3 \to C_3 + C_2]$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 3y & 3x & 1 \\ -3z & 3z+3x+9xz & 1+3z \end{vmatrix} \quad \text{[by performing } C_2 \to C_2 + (3x)C_3]$$

$$= 3y(3z + 3x + 9xz) + 9xz \quad \text{[expanding along } R_1]$$

$$= 9yz + 9xy + 27xyz + 9xz$$

$$= 9[3xyz + xy + yz + zx].$$

14. If $(x^2 + y^2)^2 = xy$, find $\dfrac{dy}{dx}$.

Or

If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\dfrac{dy}{dx}$ when $\theta = \dfrac{\pi}{3}$.

Sol. Consider

$$(x^2 + y^2)^2 = xy$$

$$\frac{d}{dx}(x^2+y^2)^2 = \frac{d}{dx}(xy)$$

$$\Rightarrow \quad 2(x^2+y^2)\left(2x + 2y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad 4x(x^2+y^2) + 4y(x^2+y^2)\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad \{4y(x^2+y^2) - x\}\frac{dy}{dx} = y - 4x(x^2+y^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - 4x(x^2+y^2)}{4y(x^2+y^2) - x}$$

Or

Consider $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$

$$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta) \quad \text{and} \quad \frac{dy}{d\theta} = a(2\sin 2\theta)$$

$$= 2a(1 - \cos 2\theta) \qquad = 2a \sin 2\theta$$

$$= 2a(2\sin^2 \theta)$$

$$= 4a \sin^2\theta$$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{2a \sin 2\theta}{4a \sin^2\theta} = \frac{2 \sin\theta \cos\theta}{2 \sin^2\theta} = \cot\theta$

$\therefore \quad \left.\frac{dy}{dx}\right]_{\theta = \frac{\pi}{3}} = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}.$

15. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0.$

Sol. Consider $\quad y = \sin(\sin x) \quad$ ...(*i*)

$\Rightarrow \quad \frac{dy}{dx} = \cos(\sin x).\cos x \quad$ ...(*ii*)

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \cos x[-\sin(\sin x).\cos x] - \sin x.\cos(\sin x)$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -y \cdot \cos^2 x - \frac{\sin x}{\cos x}.\cos x.\cos(\sin x) \quad$ [from (*i*)]

$\Rightarrow \quad \frac{d^2y}{dx^2} = -y \cdot \cos^2 x - \tan x\left(\frac{dy}{dx}\right) \quad$ [from (*ii*)]

$\Rightarrow \quad \frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y \cdot \cos^2 x = 0.$

16. Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$.

Or

Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is (*a*) strictly increasing (*b*) strictly decreasing.

Sol. Given curve is $16x^2 + 9y^2 = 145 \quad$ ...(*i*)

When $x_1 = 2$,

$$16 \times 4 + 9y^2 = 145 \Rightarrow 9y^2 = 145 - 64 = 81$$

$\Rightarrow \quad y^2 = 9 \Rightarrow y = 3 \quad (\because y_1 > 0)$

$\therefore$ Point (x_1, y_1) is (2, 3)

Differentiating (*i*) w.r.t. x, we get

$$32x + 18y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{16x}{9y}$$

$$\left.\frac{dy}{dx}\right]_{(2,\,3)} = \frac{-16\times 2}{9\times 3} = -\frac{32}{27} \qquad \text{(slope of tangent)}$$

$\therefore$ equation of tangent at (2, 3) is

$$y - 3 = -\frac{32}{27}(x-2) \Rightarrow 27y - 81 = -32x + 64$$

$$\Rightarrow \quad 32x + 27y - 145 = 0$$

$$\text{Slope of normal} = \frac{27}{32} \qquad \left[\because \text{ slope of normal} = \frac{-1}{\text{slope of tangent}}\right]$$

$\therefore$ equation of normal at (2, 3) is

$$y - 3 = \frac{27}{32}(x-2) \Rightarrow 32y - 96 = 27x - 54$$

$$\Rightarrow \quad 27x - 32y + 42 = 0.$$

Or

Consider

$$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$$

$$f'(x) = x^3 - 3x^2 - 10x + 24$$

We notice for $x = 2$, $f'(x) = 0$

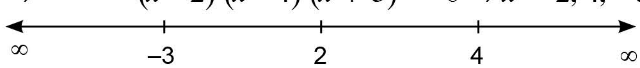
$$f'(x) = (x-2)(x^2 - x - 12)$$

$$f'(x) = (x-2)(x-4)(x+3)$$

$$\begin{array}{r|l} & x^2 - x - 12 \\ x - 2 & x^3 - 3x^2 - 10x + 24 \\ & \underset{-\;\;+}{x^3 - 2x^2} \\ \hline & -x^2 - 10x \\ & \underset{+\;\;-}{-x^2 + 2x} \\ \hline & -12x + 24 \\ & \underset{+\;\;-}{-12x + 24} \\ \hline & 0 \end{array}$$

For stationary points $f'(x) = 0$

$$\Rightarrow \quad (x-2)(x-4)(x+3) = 0 \Rightarrow x = 2, 4, -3$$



	$x < -3$	$-3 < x < 2$	$2 < x < 4$	$x > 4$
$(x-2)$	–	–	+	+
$(x-4)$	–	–	–	+
$(x+3)$	–	+	+	+
$f'(x)$	–	+	–	+
	↓	↑	↓	↑

$\therefore$ function is strictly increasing for $(-3, 2) \cup (4, \infty)$

function is strictly decreasing for $(-\infty, -3) \cup (2, 4)$

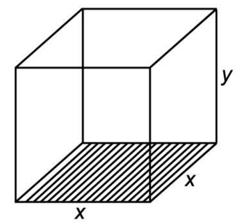
17. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question?

Sol. Let x be the side of the square and y be vertical side of the tank

Volume of tank, $V = x^2y$ ...(*i*)

Surface area, $S = 4xy + x^2$

$\Rightarrow S = 4x \cdot \frac{V}{x^2} + x^2 = \frac{4V}{x} + x^2$ [from (*i*)]



$\frac{dS}{dx} = \frac{-4V}{x^2} + 2x$, For minimum S, $\frac{dS}{dx} = 0 \Rightarrow x^3 = 2V$

$\frac{d^2S}{dx^2} = \frac{8V}{x^3} + 2$

$$\left.\frac{d^2S}{dx^2}\right]_{x^3 = 2V} = \frac{8V}{2V} + 2 > 0$$

Hence, surface area is minimum at $x^3 = 2V$

$\Rightarrow x^3 = 2x^2y \Rightarrow x = 2y \Rightarrow y = \frac{1}{2}x,$

i.e. depth $= \frac{1}{2}$ width.

Concern for others and the families should also feel that tank is their's and take care of tank.

18. Find $\int \frac{2 \cos x}{(1 - \sin x)(1 + \sin^2 x)} dx$

Sol. Consider $\int \frac{2\cos x}{(1 - \sin x)(1 + \sin^2 x)} dx$ | Let $\sin x = t$ $\Rightarrow \cos x\, dx = dt$

$$= 2\int \frac{1}{(1-t)(1+t^2)} dt \quad ...(i)$$

Let $\frac{1}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt + C}{1 + t^2}$

$\Rightarrow$ $1 = A(1 + t^2) + (Bt + C)(1 - t)$

$\Rightarrow$ $1 = A + At^2 + Bt - Bt^2 + C - Ct = t^2(A - B) + t(B - C) + (A + C)$

Comparing the coefficients, we get

$$A - B = 0, B - C = 0, A + C = 1$$

$\Rightarrow$ $A = B = C = \frac{1}{2}$

$\therefore \int \frac{1}{(1-t)(1+t^2)} dt = \frac{1}{2}\int \frac{1}{1-t} dt + \frac{1}{2}\int \frac{t+1}{t^2+1} dt$

$$= \frac{1}{2}\int \frac{1}{1-t}dt + \frac{1}{4}\int \frac{2t}{t^2+1}dt + \frac{1}{2}\int \frac{1}{t^2+1}dt$$

$$= -\frac{1}{2}\log|1-t| + \frac{1}{4}\log|t^2+1| + \frac{1}{2}\tan^{-1}t$$

$\therefore$ From (*i*)

$$\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx = 2\left[-\frac{1}{2}\log|1-t| + \frac{1}{4}\log|t^2+1| + \frac{1}{2}\tan^{-1}t\right] + C$$

$$= -\log|1-\sin x| + \frac{1}{2}\log|\sin^2 x + 1| + \tan^{-1}(\sin x) + C$$

19. Find the particular solution of the differential equation

$e^x \tan y\, dx + (2 - e^x)\sec^2 y\, dy = 0$, given that $y = \frac{\pi}{4}$ when $x = 0$.

Or

Find the particular solution of the differential equation $\frac{dy}{dx} + 2y\tan x = \sin x$, given that $y = 0$ when $x = \frac{\pi}{3}$.

Sol. Consider equation $e^x \tan y\, dx + (2 - e^x)\sec^2 y\, dy = 0$

$\Rightarrow \quad (2 - e^x)\sec^2 y\, dy = -e^x \tan y\, dx$

$\Rightarrow \quad \frac{\sec^2 y}{\tan y}dy = \frac{e^x}{e^x - 2}dx$

Integrating both sides, we get

$$\int \frac{\sec^2 y}{\tan y}dy = \int \frac{e^x}{e^x - 2}dx$$

$\Rightarrow \quad \log|\tan y| = \log|e^x - 2| + \log C$

$= \log|C(e^x - 2)|$

$\Rightarrow \quad \tan y = C(e^x - 2) \qquad ...(i)$

Given $y = \frac{\pi}{4}$, when $x = 0$

$\Rightarrow \quad \tan\frac{\pi}{4} = C(e^0 - 2) \Rightarrow 1 = -C \Rightarrow C = -1$

Substituting in (*i*), we get

$\tan y = -(e^x - 2)$ or $\tan y = 2 - e^x$ is particular solution.

Or

Consider equation $\dfrac{dy}{dx} + 2y\tan x = \sin x$

Here I.F. $= e^{2\int \tan x\, dx} = e^{2\log|\sec x|} = \sec^2 x$

Solution is $\sec^2 x \cdot y = \int \dfrac{\sin x}{\cos^2 x} dx = \int \sec x \tan x\, dx \Rightarrow \sec^2 x \cdot y = \sec x + C$

$\Rightarrow \quad y = \cos x + C\cos^2 x$

$\Rightarrow \quad y = \cos x + C.\cos^2 x \quad ...(i)$

Given $y = 0$, when $x = \dfrac{\pi}{3}$

$\Rightarrow \quad 0 = \cos\dfrac{\pi}{3} + C.\cos^2\dfrac{\pi}{3} \Rightarrow 0 = \dfrac{1}{2} + \dfrac{C}{4} \Rightarrow C = -2$

Substituting in (i), we get

$y = \cos x - 2\cos^2 x$ is particular solution.

20. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}\,.\,\vec{a} = 21$.

Sol. Given $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$

$\vec{d}$ is perpendicular to $\vec{c}$ and $\vec{b}$

$\Rightarrow \vec{d}$ is parallel to vector $\vec{c} \times \vec{b} \Rightarrow \vec{d} = \lambda(\vec{c} \times \vec{b}) \quad ...(i)$

$$\vec{c} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -1 \\ 1 & -4 & 5 \end{vmatrix} = \hat{i}(1) - \hat{j}(16) + \hat{k}(-13) = \hat{i} - 16\hat{j} - 13\hat{k}$$

$\therefore \quad \vec{d} = \lambda(\hat{i} - 16\hat{j} - 13\hat{k}) \quad$ [from (i)] ...(ii)

Also $\quad \vec{d} \cdot \vec{a} = 21 \Rightarrow 4(\lambda) + 5(-16\lambda) - 1(-13\lambda) = 21$

$\Rightarrow \quad 4\lambda - 80\lambda + 13\lambda = 21 \Rightarrow -63\lambda = 21 \Rightarrow \lambda = -\dfrac{1}{3}$

From (ii), we get

$$\vec{d} = \left(-\frac{1}{3}\hat{i} + \frac{16}{3}\hat{j} + \frac{13}{3}\hat{k}\right).$$

21. Find the shortest distance between the lines $\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$.

Sol. Consider lines $\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$,

Here $\vec{a_1} = 4\hat{i} - \hat{j}$, $\vec{b_1} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{a_2} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b_2} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

$$\vec{a_2} - \vec{a_1} = \hat{i} - \hat{j} + 2\hat{k} - 4\hat{i} + \hat{j} = -3\hat{i} + 2\hat{k}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = \hat{i}(2) - \hat{j}(1) + \hat{k}(0) = 2\hat{i} - \hat{j}$$

$$\text{The shortest distance} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| = \left| \frac{(-3\hat{i} + 2\hat{k}) \cdot (2\hat{i} - \hat{j})}{\sqrt{4+1}} \right|$$

$$= \left| \frac{-6 + 0 + 0}{\sqrt{5}} \right| = \frac{6}{\sqrt{5}} \text{ units.}$$

22. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die?

Sol. A : getting 1 or 2 ; B : getting 3, 4, 5 or 6

$P(A) = \frac{2}{6} = \frac{1}{3}$; $P(B) = \frac{4}{6} = \frac{2}{3}$

E : obtained exactly 1 tail

$P(E/A) = \frac{3}{8}$ [$\because$ a coin is tossed three times, for exactly one tail *THH, HTH, HHT*]

$P(E/B) = \frac{1}{2}$ [a coin is tossed once]

Using Bayes' Theorem, probability of obtaining a tail; when she throw 3, 4, 5 or 6 with a die.

$$P(B/E) = \frac{P(B) \cdot P(E/B)}{P(A) \cdot P(E/A) + P(B) \cdot P(E/B)} = \frac{\frac{2}{3} \cdot \frac{1}{2}}{\frac{1}{3} \cdot \frac{3}{8} + \frac{2}{3} \cdot \frac{1}{2}} = \frac{8}{3+8} = \frac{8}{11}.$$

23. Two numbers are selected at random (without replacement) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mean and variance of X.

Sol. First five positive integers are 1, 2, 3, 4, 5.

Total possibilities for two numbers are (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5), i.e. 10

X : larger of two numbers

$\therefore$ X can take values 2, 3, 4, 5

For $X = 2$, favourable possibilities (1, 2)

$X = 3$, favourable possibilities (1, 3), (2, 3)

$X = 4$, favourable possibilities (1, 4), (2, 4), (3, 4)

$X = 5$, favourable possibilities (1, 5), (2, 5), (3, 5), (4, 5)

Table for probability distribution and calculation of mean and variance:

X	$P(X)$	$X \cdot P(X)$	$X^2 \cdot P(X)$
2	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
3	$\frac{2}{10}$	$\frac{6}{10}$	$\frac{18}{10}$
4	$\frac{3}{10}$	$\frac{12}{10}$	$\frac{48}{10}$
5	$\frac{4}{10}$	$\frac{20}{10}$	$\frac{100}{10}$
		$\Sigma X \cdot P(X) = \frac{40}{10} = 4$	$\Sigma X^2 \cdot P(X) = \frac{170}{10} = 17$

$$\therefore \quad \text{Mean} = \Sigma X \cdot P(X) = 4$$

$$\text{Variance} = \Sigma X^2 \cdot P(X) - \{\Sigma X \cdot P(X)\}^2$$

$$= 17 - (4)^2$$

$$= 17 - 16 = 1$$

24. Let $A = \{x \in Z : 0 \leq x \leq 12\}$. Show that $R = \{(a, b) : a, b \in A, |a - b|$ is divisible by 4$\}$ is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2].

Or

Show that the function $f: R \to R$ defined by $f(x) = \frac{x}{x^2+1}$, $\forall\, x \in R$ is neither one-one nor onto. Also, if $g: R \to R$ is defined as $g(x) = 2x - 1$, find $fog(x)$.

Sol. Given set $A = \{x \in Z : 0 \le x \le 12\}$

$R = \{(a, b) : a, b \in A \mid a - b \mid \text{ is divisible by } 4\}$

For reflexive:

For $a \in A$, $(a, a) \in R$

$\Rightarrow$ $| a - a |$ is divisible by 4

$\Rightarrow$ 0 is divisible by 4, true.

Hence, reflexive.

For symmetric:

For $a, b \in A$, $(a, b) \in R$

$\Rightarrow$ $| a - b |$ is divisible by 4

$\Rightarrow$ $| -(b - a) |$ is divisible by 4

$\Rightarrow$ $| b - a |$ is divisible by 4 $\Rightarrow (b, a) \in R$

As $(a, b) \in R \Rightarrow (b, a) \in R$, for $a, b \in A$

Hence, symmetric.

For transitive:

For $a, b, c \in A$

Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $|a - b|$ is divisible by 4 and $| b - c |$ is divisible by 4

$\Rightarrow$ $(a - b)$ is divisible by 4 and $(b - c)$ is divisible by 4 ...(i)

$\Rightarrow$ $(a - b) + (b - c) = \text{a} - \text{c}$ is divisible by 4 [using (i)]

$\Rightarrow$ $| a - c |$ is divisible by 4.

$\Rightarrow$ $(a, c) \in R$. Hence, transitive.

As the relation R is reflexive, symmetric and transitive. Hence, relation R is an equivalence relation.

If $a \in A$ is related to 1, then $(a, 1) \in R$

$\Rightarrow$ $|a - 1|$ is divisible by 4

$\Rightarrow$ $a - 1 = 4\lambda$, for $\lambda \in N$.

$\Rightarrow \quad a = 1 + 4\lambda \quad \Rightarrow \quad a = 1, 5, 9$

Set of elements related to 1 is {1, 5, 9}.

For equivalence class [2] $\Rightarrow |a - 2|$ is divisible by 4

$\Rightarrow \quad a - 2 = 4\lambda \Rightarrow a = 2 + 4\lambda \Rightarrow a = 2, 6, 10$

$\therefore$ Equivalence class [2] = {2, 6, 10}.

Or

For one-one: For $x_1, x_2 \in R$

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow \quad x_1x_2^2 + x_1 = x_1^2x_2 + x_2 \quad \Rightarrow \quad x_1x_2(x_2 - x_1) + (x_1 - x_2) = 0$$

$$\Rightarrow \quad (x_2 - x_1)(x_1x_2 - 1) = 0 \quad \Rightarrow \quad x_2 - x_1 = 0 \text{ or } x_1x_2 = 1$$

$$\Rightarrow \quad x_1 = x_2 \text{ or } x_1x_2 = 1$$

Let $x_1 = 2$ and $x_2 = \frac{1}{2}$, then we notice $f(x_1) = f(x_2)$ but $2 \neq \frac{1}{2}$. Hence, not one-one

Here, we notice $f(x) \neq 1$ for any $x \in R$

Therefore, $1 \in R$ from co-domain does not have pre-image in domain. So, not onto.

Further $$f(x) = \frac{x}{x^2+1} \text{ and } g(x) = 2x - 1,$$

$$(fog)(x) = f[g(x)] = f(2x-1)$$

$$= \frac{2x-1}{(2x-1)^2+1} = \frac{2x-1}{4x^2-4x+2}$$

25. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Use it to solve the system of equations

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3.$$

Or

Using elementary row transformations, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}.$$

Sol. $$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2(0) + 3(-2) + 5(1) = -1 \neq 0$$

$$\text{Adj}\,A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = -\frac{1}{1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \qquad ...(i)$$

Consider equations,

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Corresponding matrix equation is

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

i.e. $AX = B$ is matrix. Its solution is $X = A^{-1}B$

$$X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \qquad \text{[from } (i)]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ $x = 1, y = 2, z = 3$ is solution.

Or

Consider $A = IA$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} A \quad \text{[by performing } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_3 + R_2]$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A \quad \text{[by performing } R_3 \to R_3 - R_2]$$

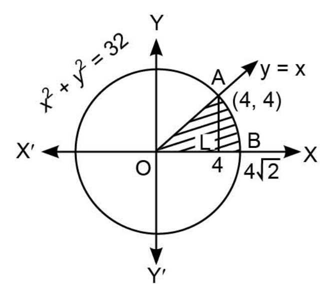
$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}5 & -2 & 0\\-2 & 1 & 0\\2 & 0 & 1\end{bmatrix}A \qquad \text{[by performing } R_1 \to R_1 - 2R_2]$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}3 & -2 & -1\\-4 & 1 & -1\\2 & 0 & 1\end{bmatrix}A \quad \text{[by performing } R_1 \to R_1 - R_3 \text{ and } R_2 \to R_2 - R_3]$$

$$\therefore \qquad A^{-1} = \begin{bmatrix}3 & -2 & -1\\-4 & 1 & -1\\2 & 0 & 1\end{bmatrix}$$

26. Using integration, find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

Sol. $x^2 + y^2 = 32$ is a circle with centre (0, 0), radius $4\sqrt{2}$, $y = x$ is a line passing through (0, 0)



Eliminating y from $x^2 + y^2 = 32$ and $x = y$, we get

$2x^2 = 32 \Rightarrow x = 4$ (for first quadrant)

ar(OAB) = ar(OAL) + ar(LAB)

ar(OAB) : area bounded by the curve $y = x$, the x-axis and between $x = 0$ and $x = 4$

ar(LAB) : area bounded by the curve $x^2 + y^2 = 32$ the x-axis and between $x = 4$ and $x = 4\sqrt{2}$

$$\therefore \text{ Area } (OAB) = \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{32 - x^2} + \frac{32}{2}\sin^{-1}\frac{x}{\sqrt{32}}\right]_4^{4\sqrt{2}}$$

$$= \left(\frac{16}{2} - 0\right) + \left(\frac{4\sqrt{2}}{2} \times 0 + 16\sin^{-1}1\right) - \left(\frac{4}{2}\sqrt{32 - 16} + 16\sin^{-1}\frac{1}{\sqrt{2}}\right)$$

$$= 8 + 16 \times \frac{\pi}{2} - 2 \times 4 - 16 \times \frac{\pi}{4} = 8\pi - 4\pi = 4\pi \text{ sq units}$$

27. Evaluate $\displaystyle\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x}\,dx$

Or

Evaluate $\displaystyle\int_1^3 (x^2 + 3x + e^x)\,dx$**, as the limit of the sum.**

Sol. $\int \frac{\sin x + \cos x}{25 - 9(1 - \sin 2x)} dx = \int \frac{\sin x + \cos x}{25 - 9(\cos x - \sin x)^2} dx$

Let $\cos x - \sin x = t$
$\Rightarrow (-\sin x - \cos x)\, dx = dt$
$\Rightarrow (\sin x + \cos x) dx = -dt$
When $x = 0, t = 1$
When $x = \frac{\pi}{4}, t = 0$

$$= -\int \frac{1}{25 - 9t^2} dt = -\frac{1}{9}\int \frac{1}{\left(\frac{5}{3}\right)^2 - t^2} dt$$

$$= -\frac{1}{9} \times \frac{1}{2 \times \frac{5}{3}} \log \left| \frac{\frac{5}{3} + t}{\frac{5}{3} - t} \right| = -\frac{1}{30} \log \left| \frac{5 + 3t}{5 - 3t} \right|$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{16 + 9 \sin 2x} dx = -\frac{1}{30}\left[\log\left|\frac{5 + 3t}{5 - 3t}\right|\right]_1^0$$

$$= -\frac{1}{30}[\log 1 - \log 4] = \frac{1}{30} \log 4$$

Or

Consider $\int_1^3 (x^2 + 3x + e^x) dx$

Here, $a = 1, b = 3, f(x) = x^2 + 3x + e^x, h = \frac{3 - 1}{n} \Rightarrow nh = 2$ ...(*i*)

$$\int_1^3 (x^2 + 3x + e^x) dx = \lim_{h \to 0} h[f(1) + f(1 + h) + f(1 + 2h) + \ldots + f\{1 + (n - 1)h\}]$$ ...(*ii*)

$$f(1) = 1 + 3 + e = 4 + e$$
$$f(1 + h) = (1 + h)^2 + 3(1 + h) + e^{1 + h} = h^2 + 5h + 4 + e^{1+h}$$
$$f(1 + 2h) = (1 + 2h)^2 + 3(1 + 2h) + e^{1 + 2h} = 4h^2 + 10h + 4 + e^{1+2h}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$f\{1 + (n - 1)h\} = \{1 + (n - 1)h\}^2 + 3\{1 + (n - 1)h\} + e^{1 + (n - 1)h}$$
$$= (n - 1)^2h^2 + 5(n - 1)h + 4 + e^{1 + (n - 1)h}$$

From (*ii*)

$$\int_1^3 (x^2 + 3x + e^x) dx = \lim_{h \to 0} h[(4 + e) + (h^2 + 5h + 4 + e^{1 + h}) + (4h^2 + 10h + 4 + e^{1 + 2h}) + \ldots + (n - 1)^2h^2 + 5(n - 1)h + 4 + e^{1 + (n - 1)h}$$

$$= \lim_{h \to 0} h[h^2\{1 + 4 + \ldots + (n - 1)^2\} + 5h\{1 + 2 + \ldots + (n - 1)\} + 4n + \{e + e^{1 + h} + e^{1 + 2h} + \ldots + e^{1 + (n - 1)h}\}]$$

$$= \lim_{h \to 0} h\left[h^2 \cdot \frac{(n - 1)\, n\,(2n - 1)}{6} + 5h \cdot \frac{(n - 1)n}{2} + 4n + e\left\{\frac{1\{(e^h)^n - 1\}}{e^h - 1}\right\}\right]$$

$$= \lim_{h\to 0}\left[\frac{(nh-h)(nh)(2nh-h)}{6} + \frac{5}{2}(nh-h)(nh) + 4nh + e\cdot\frac{h}{e^h-1}\cdot\{e^{nh}-1\}\right]$$

$$= \lim_{h\to 0}\left[\frac{(2-h)(2)(4-h)}{6} + \frac{5}{2}(2-h)(2) + 4\times 2 + e\cdot\frac{h}{e^h-1}\{e^2-1\}\right]$$

$$= \frac{2\times 2\times 4}{6} + \frac{5}{2}\times 2\times 2 + 8 + e(e^2-1)\cdot\lim_{h\to 0}\frac{h}{e^h-1}$$

$$= \frac{8}{3} + 10 + 8 + e(e^2-1)\times 1 = \frac{62}{3} + (e^3-e).$$

28. Find the distance of the point (–1, –5, –10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i} - \hat{j} + \hat{k}) = 5$.

Sol. General point on the line is $\vec{r} = (2+3\lambda)\hat{i} + (-1+4\lambda)\hat{j} + (2+2\lambda)\hat{k}$

If point lies on the plane, $\vec{r}\cdot(\hat{i} - \hat{j} + \hat{k}) = 5$

then $1(2 + 3\lambda) - 1(-1 + 4\lambda) + 1(2 + 2\lambda) = 5 \Rightarrow \lambda = 0$

$\therefore$ point of intersection is (2, – 1, 2).

Distance $= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$ units

29. A factory manufactures two types of screws *A* and *B*, each type requiring the use of two machines, an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machines to manufacture a packet of screws '*A*' while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a packet of screws '*B*'. Each machine is available for at most 4 hours on any day. The manufacturer can sell a packet of screws '*A*' at a profit of 70 paise and screws '*B*' at a profit of ₹ 1. Assuming that he can sell all the screws he manufactures, how many packets of each type should the factory owner produce in a day in order to maximise his profit ? Formulate the above *LPP* and solve it graphically and find the maximum profit.

Sol. Let x packets of type A and y packets of type B are produced.

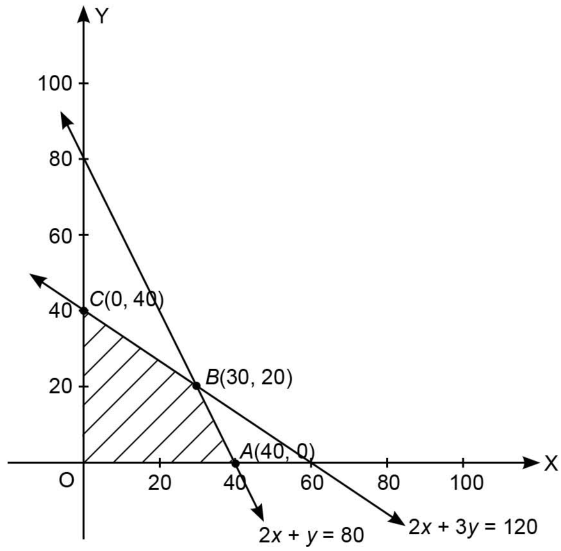
	Automatic	**Hand-operated**	**Profit**
Screws *A*	4 min	6 min	0.70 ₹
Screws *B*	6 min	3 min	1.00 ₹
	$\leq$ 4 hr	$\leq$ 4 hr	

LPP is to maximise $Z = 0.7x + y$ subject to constraints

$4x + 6y \leq 240 \Rightarrow 2x + 3y \leq 120$

$6x + 3y \leq 240 \Rightarrow 2x + y \leq 80$

$x \geq 0, y \geq 0$



On plotting the graph of inequations we notice shaded portion is feasible solution.

Possible points of maximum Z are

$A(40, 0)$, $B(30, 20)$ and $C(0, 40)$.

Points	$Z = 0.7x + y$	Values	
$A(40, 0)$	28 + 0	28	
$B(30, 20)$	21 + 20	41	←Maximum
$C(0, 40)$	0 + 40	40	

Z is maximum at $B(30, 20)$, i.e. $x = 30$, $y = 20$. Hence, 30 packets of type *A* and 20 packets of type *B* must be produced for a maximum profit of ₹ 41.

Examination Papers, 2017
[Delhi Set-I, II, III]

General Instructions:

(*i*) All questions are compulsory.

(*ii*) This question paper contains 29 questions

(*iii*) Questions **1-4** in Section A are very short-answer type questions carrying 1 mark each.

(*iv*) Questions **5-12** in Section B are short-answer type questions carrying 2 marks each.

(*v*) Questions **13-23** in Section C are long-answer I type questions carrying 4 marks each.

(*vi*) Questions **24-29** in Section D are long-answer II type questions carrying 6 marks each.

SECTION – A

Question numbers **1** *to* **4** *carry* **1** *mark each.*

1. If A is a 3×3 invertible matrix, then what will be the value of k if $\det(A^{-1}) = (\det A)^k$.

Sol.
$$\det(A^{-1}) = (\det A)^k \Rightarrow \frac{1}{|A|} = (|A|)^k$$
$$\Rightarrow \quad (|A|)^{k+1} = 1 \Rightarrow k + 1 = 0$$
$$\Rightarrow \quad k = -1.$$

2. Determine the value of the constant 'k' so that the function $f(x) = \begin{cases} \dfrac{kx}{|x|} & \text{, if } x < 0 \\ 3 & \text{, if } x \geq 0 \end{cases}$ is continuous at $x = 0$.

Sol. If continuous at $x = 0$,
$$\underset{x=0}{\text{LHL}} = \underset{x=0}{\text{RHL}} = f(0)$$
$$\Rightarrow \quad \lim_{x \to 0^-} \frac{kx}{|x|} = \lim_{x \to 0^+} (3) = 3$$
$$\Rightarrow \quad \frac{-kx}{x} = 3 = 3 \Rightarrow k = -3$$

3. Evaluate: $\int_2^3 3^x dx$.

Sol. Consider $\int_2^3 3^x dx = \left[\dfrac{3^x}{\log_e 3}\right]_2^3 = \dfrac{3^3}{\log_e 3} - \dfrac{3^2}{\log_e 3} = \log_3 e\,(27 - 9) = 18 \log_3 e.$

4. If a line makes angles 90° and 60° respectively with the positive directions of x and y axes, find the angle which it makes with the positive direction of z-axis.

Sol. Let angle with z-axis be γ.

$$\therefore \quad \cos^2 90° + \cos^2 60° + \cos^2 \gamma = 1$$

$$\Rightarrow \quad 0 + \frac{1}{4} + \cos^2 \gamma = 1 \quad \Rightarrow \quad \cos^2 \gamma = \frac{3}{4}$$

$$\Rightarrow \quad \cos \gamma = \pm\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \gamma = 30°, 150°$$

SECTION – B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

5. Show that all the diagonal elements of a skew symmetric matrix are zero.

Sol. If $A = [a_{ij}]$ is a skew symmetric matrix, then $a_{ij} = -a_{ji}, \forall i, j$

Let $i = j \quad \Rightarrow \quad a_{ii} = -a_{ii} \quad \Rightarrow \quad 2a_{ii} = 0 \quad \Rightarrow \quad a_{ii} = 0$

$\Rightarrow$ All the diagonal elements of a skew symmetric matrix are always zero.

6. Find $\frac{dy}{dx}$ at $x = 1, y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.

Sol. Consider $\sin^2 y + \cos(xy) = K$

Differentiating both sides w.r.t. x, we get

$$2 \sin y \cos y \cdot \frac{dy}{dx} - \sin(xy) \cdot \left\{x\frac{dy}{dx} + y.1\right\} = 0$$

$$\Rightarrow \quad \sin 2y \frac{dy}{dx} - x \sin(xy) \frac{dy}{dx} - y \cdot \sin(xy) = 0$$

$$\Rightarrow \quad \frac{dy}{dx}\{\sin 2y - x \sin(xy)\} = y \sin(xy)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y \sin(xy)}{\sin 2y - x \sin(xy)}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=1,\, y=\frac{\pi}{4}} = \frac{\frac{\pi}{4} \cdot \sin\frac{\pi}{4}}{\sin\frac{\pi}{2} - 1 \cdot \sin\frac{\pi}{4}}$$

$$= \frac{\frac{\pi}{4} \cdot \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{\pi}{4(\sqrt{2} - 1)}.$$

7. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm.

Sol. Let V be volume, S surface area and r radius of a sphere at any time t.

$$\frac{dV}{dt} = 3 \text{ cm}^3/\text{s. To find } \left.\frac{dS}{dt}\right]_{r=2} = ?$$

Volume of the sphere, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 . \frac{dr}{dt}$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{3}{4\pi r^2} \qquad \text{...}(i)$$

Now, surface area, $S = 4\pi r^2$

$$\Rightarrow \quad \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{3}{4\pi r^2} \qquad [\text{from } (i)]$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{6}{r}$$

$$\therefore \quad \left.\frac{dS}{dt}\right]_{r=2} = \overset{6}{\underset{2}{C}} = 3 \text{ cm}^2/\text{s}$$

Hence, surface area of the sphere is increasing at the rate of 3 cm^2/s.

8. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R.

Sol. Consider the function, $f(x) = 4x^3 - 18x^2 + 27x - 7$

$$f'(x) = 12x^2 - 36x + 27 = 12\left(x^2 - 3x + \frac{9}{4}\right)$$

$$= 12\left\{\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{9}{4}\right\} = 12\left(x - \frac{3}{2}\right)^2$$

$\therefore f'(x) \geq 0$, for $x \in R$. Hence, always increasing on R.

9. Find the vector equation of the line passing through the point $A(1, 2, -1)$ and parallel to the line $5x - 25 = 14 - 7y = 35z$.

Sol. Given line is $5x - 25 = 14 - 7y = 35z$

$\Rightarrow \quad 5(x - 5) = -7(y - 2) = 35z$

$$\Rightarrow \quad \frac{x-5}{7} = \frac{y-2}{-5} = \frac{z-0}{1}$$

DR's of line are 7, – 5 and 1

$\therefore$ DR's of line parallel to the given line are 7, – 5, 1.

$\therefore$ Vector equation of line through the points (1, 2, – 1) and having DR's 7, – 5 and1 is

$$\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k}).$$

10. Prove that if E and F are independent events, then the events E and F' are also independent.

Sol. As E and F are independent events

$\therefore \quad P(E \cap F) = P(E)P(F) \quad \ldots(i)$

Consider,

$$\begin{aligned} P(E)P(F') &= P(E)[1 - P(F)] \\ &= P(E) - P(E)P(F) \\ &= P(E) - P(E \cap F) \\ P(E)P(F') &= P(E \cap F') \end{aligned}$$

$\Rightarrow$ E and F' are independent events.

11. A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300. Formulate an L.P.P. for finding how many of each should be produced daily to maximise the profit? It is being given that at least one of each must be produced.

Sol. Let x necklaces and y bracelets be produced.

Then L.P.P. is

To maximise profit $\quad Z = 100x + 300y$

Subject to constraints $\quad x \geq 1, y \geq 1$

$$x + y \leq 24; \ \frac{1}{2}x + y \leq 16$$

12. Find $\int \frac{dx}{x^2 + 4x + 8}$

Sol. Consider

$$\int \frac{dx}{x^2 + 4x + 8} = \int \frac{dx}{(x+2)^2 + 4}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$$

SECTION – C

Question numbers **13** *to* **23** *carry* **4** *marks each.*

13. Prove that $\tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}$.

Sol. LHS $= \tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right)$

$$= \frac{1 + \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1 - 1.\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} + \frac{1 - \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1 + 1.\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} \qquad \left[\tan\left(\frac{\pi}{4} \pm \theta\right) = \frac{1 \pm \tan\theta}{1 \mp \tan\theta}\right]$$

$$= \frac{\left[1+\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right]^2 + \left[1-\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right]^2}{1-\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} = \frac{2\left[1+\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right]}{1-\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{2}{\cos 2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} = \frac{2}{\cos\left(\cos^{-1}\frac{a}{b}\right)} = \frac{2}{\frac{a}{b}} = \frac{2b}{a} = \text{RHS}\left[\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

14. Using properties of determinants, prove that

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$$

Or

Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ **find a matrix** D **such that** $CD - AB = O$

Sol. Consider, LHS $= \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

$$= \begin{vmatrix} 3x+3y & 3x+3y & 3x+3y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} \quad \text{[by performing } R_1 \to R_1 + (R_2 + R_3)]$$

$$= 3(x+y)\begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} \quad \text{[by taking } 3(x+y) \text{ common from } R_1]$$

$$= 3(x+y)\begin{vmatrix} 0 & 0 & 1 \\ 2y & -y & x+y \\ -y & 2y & x \end{vmatrix} \quad \text{[by performing } C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3]$$

$= 3(x+y)[1(4y^2 - y^2)]$ [on expanding along R_1]

$= 9y^2(x+y)$.

Or

$$CD - AB = O \Rightarrow CD = AB$$

$$\Rightarrow C^{-1}(CD) = C^{-1}(AB) \Rightarrow (C^{-1}C)D = C^{-1}(AB)$$

$$\Rightarrow ID = C^{-1}(AB) \Rightarrow D = C^{-1}(AB) \quad \text{...}(i)$$

$$AB = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 10-7 & 4-4 \\ 15+28 & 6+16 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} \quad \text{...}(ii)$$

Also, given $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$

$$\Rightarrow \quad |C| = \begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} = 16 - 15 = 1 \neq 0 \text{ and Adj } C = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$$

$$\Rightarrow \quad C^{-1} = \frac{1}{1}\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} \qquad ...(iii)$$

Substituting from (*ii*) and (*iii*) in (*i*), we get

$$D = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 24 - 215 & 0 - 110 \\ -9 + 86 & 0 + 44 \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

15. Differentiate the function $(\sin x)^x + \sin^{-1}\sqrt{x}$ with respect to x.

Or

If $x^m y^n = (x + y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$.

Sol. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Also, let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x} \quad \Rightarrow \quad y = u + v$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ...(i)$$

Consider, $u = (\sin x)^x$

Taking log on both sides, we get

$$\log u = \log(\sin x)^x \quad \Rightarrow \quad \log u = x \log(\sin x)$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = [\log(\sin x)]\frac{d}{dx}(x) + (x)\frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\log(\sin x) \cdot 1 + x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \quad \frac{du}{dx} = (\sin x)^x\left[\log(\sin x) + \frac{x}{\sin x} \cdot \cos x\right]$$

$$\Rightarrow \quad \frac{du}{dx} = (\sin x)^x (x \cot x + \log \sin x) \qquad ...(ii)$$

Now, consider $v = \sin^{-1}\sqrt{x}$

Differentiating both sides with respect to x, we obtain

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} \Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}} \qquad ...(iii)$$

Substituting values of $\frac{du}{dx}$ from (*ii*) and $\frac{dv}{dx}$ from (*iii*) in (*i*), we get

$$\frac{dy}{dx} = (\sin x)^x (x \cot x + \log \sin x) + \frac{1}{2\sqrt{x-x^2}}.$$

Or

Taking log on both sides, we get

$$m \log x + n \log y = (m+n) \log (x+y)$$

Differentiating both sides w.r.t. x, we get

$$m \cdot \frac{1}{x} + n \cdot \frac{1}{y} \cdot \frac{dy}{dx} = (m+n) \cdot \frac{1}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{m}{x} - \frac{m+n}{x+y} = \frac{dy}{dx}\left[\frac{m+n}{x+y} - \frac{n}{y}\right]$$

$$\Rightarrow \quad \frac{mx+my-mx-nx}{x(x+y)} = \frac{dy}{dx}\left[\frac{my+ny-nx-ny}{y(x+y)}\right]$$

$$\Rightarrow \quad \frac{my-nx}{x(x+y)} = \left[\frac{my-nx}{y(x+y)}\right]\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = y \Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 1 = \frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = 0$$

16. Find $\int \frac{2x}{(x^2+1)(x^2+2)^2} dx$

Sol. Consider $\int \frac{2x}{(x^2+1)(x^2+2)^2} dx$

| Let $x^2 + 1 = t$
| $\Rightarrow \quad 2x\,dx = dt$

$$= \int \frac{1}{t(t+1)^2} dt$$

$$= \int \frac{(t+1)-t}{t(t+1)^2} dt$$

$$= \int \frac{1}{t(t+1)} dt - \int \frac{1}{(t+1)^2} dt$$

$$= \int \frac{(t+1)-t}{t(t+1)} dt - \int \frac{1}{(t+1)^2} dt$$

$$= \int \frac{1}{t}dt - \int \frac{1}{t+1}dt - \int \frac{1}{(t+1)^2}dt$$

$$= \log|t| - \log|t+1| + \frac{1}{t+1} + C$$

$$= \log|x^2+1| - \log|x^2+2| + \frac{1}{x^2+2} + C$$

17. Evaluate: $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}dx.$

Or

Evaluate: $\displaystyle\int_0^{3/2} |x \sin \pi x|\, dx.$

Sol. $$I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}dx \quad \text{...}(i)$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx \left[\text{using property } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right] \quad \text{...}(ii)$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx \quad \text{[by adding (i) and (ii)]}$$

$$\Rightarrow \quad 2I = -\pi\int_1^{-1} \frac{1}{1+t^2}dt = -\pi\left[\tan^{-1} t\right]_1^{-1}$$

Let $\cos x = t$
$\Rightarrow -\sin x\, dx = dt$
when $x = 0, t = 1$
and when $x = \pi, t = -1$

$$= -\pi\,[\tan^{-1}(-1) - \tan^{-1}(1)]$$

$$2I = -\pi\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{2} \quad \Rightarrow \quad I = \frac{\pi^2}{4}$$

Or

For $0 < x < 1, \quad x \sin \pi x > 0$;

For $1 < x < \frac{3}{2}, \quad x \sin \pi x < 0$

$$\therefore \int_0^{3/2} |x \sin \pi x|\,dx = \int_0^1 x \sin \pi x\, dx + \int_1^{3/2} (-x \sin \pi x)dx \quad \text{...}(i)$$

Now, $$\int x.\sin \pi x\, dx = x.\left(\frac{-\cos \pi x}{\pi}\right) - \int 1.\left(\frac{-\cos \pi x}{\pi}\right)dx$$

$$= -\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x$$

From(i), we get

$$\int_0^{\frac{3}{2}} |x \sin \pi x|\,dx = \left(-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right)_0^1 - \left(-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right)_1^{\frac{3}{2}}$$

$$= \left(-\frac{1}{\pi}\cos\pi + \frac{1}{\pi^2}\sin\pi\right) - (0) - \left(-\frac{3}{2\pi}\cos\frac{3\pi}{2} + \frac{1}{\pi^2}\sin\frac{3\pi}{2}\right) + \left(-\frac{1}{\pi}\cos\pi + \frac{1}{\pi^2}\sin\pi\right)$$

$$= \frac{1}{\pi} + 0 - 0 - 0 + \frac{1}{\pi^2} + \frac{1}{\pi} = \frac{2}{\pi} + \frac{1}{\pi^2} = \frac{2\pi + 1}{\pi^2}$$

18. Prove that $x^2 - y^2 = C(x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$, where C is a parameter.

Sol. Consider equation,

$$(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \quad \text{...}(i)$$

Let $y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i) $\quad v + x\frac{dv}{dx} = \frac{x^3 - 3x^3v^2}{x^3v^3 - 3x^3v} = \frac{1 - 3v^2}{v^3 - 3v}$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v = \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v} \Rightarrow \int \frac{v^3 - 3v}{1 - v^4}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{v^3}{1 - v^4}dv - 3\int \frac{v}{1 - v^4}dv = \int \frac{dx}{x} \quad \text{...}(ii)$$

For $\quad \int \frac{v^3}{1 - v^4}dv = \frac{1}{4}\int \frac{1}{1 - t}dt \quad \left|\ \text{let } v^4 = t \Rightarrow 4v^3dv = dt\right.$

$$= -\frac{1}{4}\log|1 - t| = -\frac{1}{4}\log|1 - v^4|$$

For $\quad \int \frac{v}{1 - v^4}dv = \frac{1}{2}\int \frac{1}{1 - t^2}dt \quad \left|\ \text{Let } v^2 = t \Rightarrow 2vdv = dt\right.$

$$= \frac{1}{2 \times 2}\log\left|\frac{1 + t}{1 - t}\right| = \frac{1}{4}\log\left|\frac{1 + v^2}{1 - v^2}\right|$$

From (ii) we get

$$-\frac{1}{4}\log|1 - v^4| - \frac{3}{4}\log\left|\frac{1 + v^2}{1 - v^2}\right| = \log|x| + \log|k|$$

$$\Rightarrow \quad \frac{1}{4}\left[\log|1-v^4| + 3\log\left|\frac{1+v^2}{1-v^2}\right|\right] = \log\left|\frac{1}{xk}\right|$$

$$\Rightarrow \quad \log\left|\frac{(1-v^2)(1+v^2)\times(1+v^2)^3}{(1-v^2)^3}\right| = \log\left|\frac{1}{xk}\right|^4$$

$$\Rightarrow \quad \frac{(1+v^2)^4}{(1-v^2)^2} = \frac{1}{x^4k^4} \quad \Rightarrow \quad (1+v^2)^2 = \frac{(1-v^2)}{x^2k^2}$$

$$\Rightarrow \quad \left(1+\frac{y^2}{x^2}\right)^2 x^2k^2 = \left(1-\frac{y^2}{x^2}\right)$$

$$\Rightarrow \quad \frac{(x^2+y^2)^2}{x^4}\cdot x^2k^2 = \frac{(x^2-y^2)}{x^2}$$

$\Rightarrow$ $(x^2+y^2)^2C = x^2 - y^2$ is solution (where $k^2 = C$)

19. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

(*a*) Let $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

(*b*) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

Sol. (*a*) Given $c_1 = 1$ and $c_2 = 2$

$\therefore \quad \vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = \hat{i} + 2\hat{j} + c_3\hat{k}$

If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = 0 \Rightarrow -1[c_3 - 2] = 0 \Rightarrow c_3 = 2$$

(*b*) When $c_2 = -1$, $c_3 = 1$, then if $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar then

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & -1 & 1 \end{vmatrix} = 0 \Rightarrow -1[1+1] = 0$$

$\Rightarrow -2 = 0$, not possible for any value of c_1.

So, no value of c_1 can make vectors $\vec{a}, \vec{b}, \vec{c}$ coplanar.

20. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a}+\vec{b}+\vec{c}$ makes with $\vec{a}$ or $\vec{b}$ *or* $\vec{c}$.

Sol. $|\vec{a}| = |\vec{b}| = |\vec{c}|$ and $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$ ...(*i*)

Let α be angle between $\vec{a} + \vec{b} + \vec{c}$ and $\vec{a}$

$$\cos\alpha = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} \qquad \text{[from } (i)] \text{ ...}(ii)$$

Similarly show angle between $(\vec{a}+\vec{b}+\vec{c})$ and $\vec{b}$ is $\cos\beta = \dfrac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$ ...(*iii*)

and angle between $(\vec{a}+\vec{b}+\vec{c})$ and $\vec{c}$ is $\cos\gamma = \dfrac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$ ...(*iv*)

From (*i*), (*ii*), (*iii*) and (*iv*), we get

$$\cos\alpha = \cos\beta = \cos\gamma \Rightarrow \alpha = \beta = \gamma.$$

Also $\quad |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}\,|\vec{a}|$ [from (*i*)]

$\therefore \quad \cos\alpha = \dfrac{|\vec{a}|}{\sqrt{3}\,|\vec{a}|} = \dfrac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

Hence, $\vec{a}+\vec{b}+\vec{c}$ makes angle $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ with any of vectors $\vec{a}, \vec{b}$ or $\vec{c}$.

[**Note:** $\vec{a}, \vec{b}, \vec{c}$ can be taken along sides of a cube, then $\vec{a} + \vec{b} + \vec{c}$ is along diagonal of the cube]

21. The random variable X can take only the values 0, 1, 2, 3. Given that $P(X = 0) = P(X = 1) = p$ and $P(X = 2) = P(X = 3)$ such that $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$ find the value of p.

Sol. Let $P(X = 2) = P(X = 3) = k$(say)

We have $P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1$

$\Rightarrow \quad p + p + k + k = 1 \;\Rightarrow\; 2k = 1 - 2p \;\Rightarrow\; k = \dfrac{1}{2}(1-2p)$

Distribution is

X	$P(X)$	$X \cdot P(X)$	$X^2 \cdot P(X)$
0	p	0	0
1	p	p	p
2	$\frac{1}{2}(1-2p)$	$(1-2p)$	$2(1-2p)$
3	$\frac{1}{2}(1-2p)$	$\frac{3}{2}(1-2p)$	$\frac{9}{2}(1-2p)$

Given $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$

$$\therefore \quad p + 2(1-2p) + \frac{9}{2}(1-2p) = 2\left[p + (1-2p) + \frac{3}{2}(1-2p)\right]$$

$$\Rightarrow \quad p + \frac{13}{2}(1-2p) = 2p + 5(1-2p)$$

$$\Rightarrow \quad p + \frac{13}{2} - 13p = 2p + 5 - 10p$$

$$\Rightarrow \quad p - 13p - 2p + 10p = 5 - \frac{13}{2} \Rightarrow -4p = -\frac{3}{2} \Rightarrow p = \frac{3}{8}$$

22. Often it is taken that a truthful person commands, more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Do you also agree that the value of truthfulness leads to more respect in the society?

Sol. A : getting six $\qquad$ B : not getting six

$P(A) = \frac{1}{6}$ $\qquad$ $P(B) = \frac{5}{6}$

E : reports six

$P(E/A) = \frac{4}{5}$ $\qquad$ $P(E/B) = \frac{1}{5}$

Using Baye's Theorem,

Probability that is actually a six,

$$P(A/E) = \frac{P(A)P(E/A)}{P(A)P(E/A) + P(B)P(E/B)}$$

$$= \frac{\frac{1}{6}\times\frac{4}{5}}{\frac{1}{6}\times\frac{4}{5}+\frac{5}{6}\times\frac{1}{5}} = \frac{4}{4+5} = \frac{4}{9}$$

Yes, we agree that the value of truthfulness leads to more respect in the society.

23. Solve the following L.P.P. graphically

Minimise $Z = 5x + 10y$

Subject to Constraints $x + 2y \leq 120$

$x + y \geq 60$

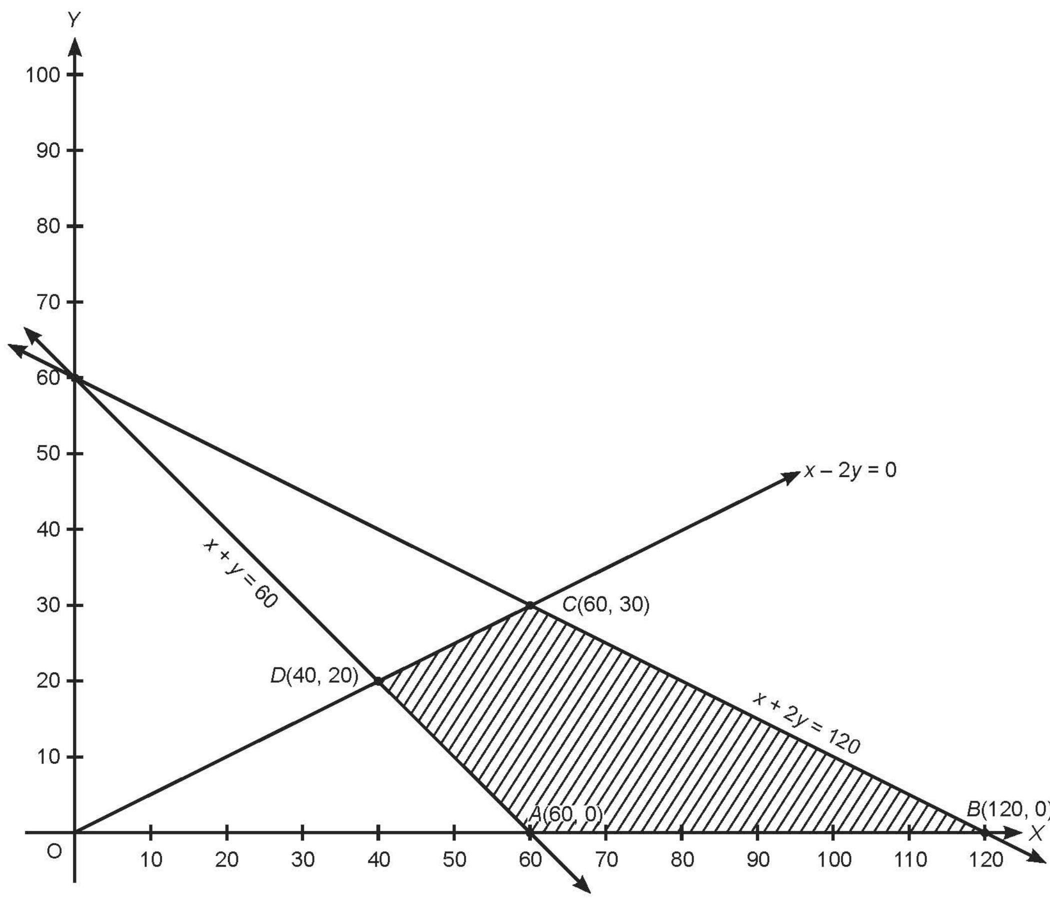
$x - 2y \geq 0$

and $x, y \geq 0$

Sol. To minimise $Z = 5x + 10y$

Subject to the constraints

$$x \geq 0, y \geq 0$$
$$x - 2y \geq 0$$
$$x + y \geq 60$$
$$x + 2y \leq 120$$



Plotting the inequations on graph, we notice shaded portion is feasible solution. Possible points for minimum Z are $A(60, 0)$, $B(120, 0)$, $C(60, 30)$ and $D(40, 20)$

Points	**$Z = 5x + 10y$**	**Value**	
$A(60, 0)$	300 + 0	300	←Minimum
$B(120, 0)$	600 + 0	600	
$C(60, 30)$	300 + 300	600	
$D(40, 20)$	200 + 200	400	

Z in minimum for $A(60, 0)$. Hence, for $x = 60$ and $y = 0$, Z is minimum.

SECTION – D

Question numbers **24** *to* **29** *carry* **6** *marks each*.

24. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ **to solve the system of equations** $x + 3z = 9$, $-x + 2y - 2z = 4$, $2x - 3y + 4z = -3$.

Sol. Consider $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \qquad AB = I \qquad$ ...(*i*)

Consider equations

$$x + 3z = 9$$
$$-x + 2y - 2z = 4$$
$$2x - 3y + 4z = -3$$

Corresponding matrix equation is

$$\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$(A^T)X = C$ is matrix equation

Its solution is $X = (A^T)^{-1}C = (A^{-1})^TC \qquad$ ...(*ii*)

From (i), we have $\quad AB = I$

$\Rightarrow \quad A^{-1} = B \Rightarrow (A^{-1})^T = B^T$

$$\Rightarrow \quad (A^{-1})^T = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}^T = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}$$

From (ii), we get

$$X = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} -18+36-18 \\ 0+8-3 \\ 9-12+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad x = 0, y = 5$ and $z = 3$ is solution.

25. Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\dfrac{\sqrt{y+6} - 1}{3}\right)$

Hence find

(i) $f^{-1}(10)$ $\qquad$ (ii) y if $f^{-1}(y) = \dfrac{4}{3}$

Where R_+ is the set of all non-negative real numbers.

Or

Discuss the commutativity and associativity of binary operation '*' defined on $A = Q - \{1\}$ by the rule $a * b = a - b + ab$ for all $a, b \in A$. Also, find the identity element of * in A and hence find the invertible elements of A.

Sol. Given $f : R_+ \to [-5, \infty)$, given by $f(x) = 9x^2 + 6x - 5$.

For one-one: Let for $x_1 x_2 \in R_+$,

$$f(x_1) = f(x_2)$$

$$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$

$$\Rightarrow \quad 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0 \Rightarrow (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$$

$$9(x_1 + x_2) + 6 \neq 0, \text{ as } x_1, x_2 \in R_+$$

$$\Rightarrow \quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

As $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Hence, one-one.

For onto: Let for $a \in [-5, \infty)$ there exists $x \in R_+$ such that $f(x) = a$

$$\Rightarrow \quad a = 9x^2 + 6x - 5 \Rightarrow 9x^2 + 6x - (5 + a) = 0.$$

$$\Rightarrow \quad x = \frac{-6 \pm \sqrt{36 + 36(5+a)}}{18} = \frac{-1 \pm \sqrt{(6+a)}}{3}$$

$$\Rightarrow \quad x = \frac{\sqrt{(6+a)} - 1}{3} \in R_+, \text{ so onto.}$$

We define a function $g : [-5, \infty) \to R_+$ as $g(x) = \dfrac{\sqrt{(6+x)} - 1}{3}$

$$fog(x) = f(g(x)) = f\left\{\frac{\sqrt{(6+x)} - 1}{3}\right\}$$

$$= \left\{9\left(\frac{\sqrt{(6+x)} - 1}{3}\right)^2 + 6\left(\frac{\sqrt{(6+x)} - 1}{3}\right) - 5\right\}$$

$$= 9\left(\frac{6 + x + 1 - 2\sqrt{(6+x)}}{9}\right) + 2\left(\sqrt{(6+x)} - 1\right) - 5$$

$$= 7 + x - 2\sqrt{(6+x)} + 2\sqrt{(6+x)} - 2 - 5 = x$$

and $\quad gof(x) = g(f(x)) = g\,(9x^2 + 6x - 5)$

$$= \frac{\sqrt{(6 + 9x^2 + 6x - 5)} - 1}{3} = \frac{\sqrt{(3x+1)^2} - 1}{3} = \frac{3x + 1 - 1}{3} = x$$

As $fog = I_{[-5, \infty)}$ and $gof = I_{R+}$

Hence, f is invertible and $f^{-1}(x) = \dfrac{\sqrt{(6+x)} - 1}{3}$ or $f^{-1}(y) = \dfrac{\sqrt{(y+6)} - 1}{3}$.

We have $\quad f^{-1}(y) = \dfrac{\sqrt{y+6} - 1}{3}$

(*i*) $\quad f^{-1}(10) = \dfrac{\sqrt{10+6} - 1}{3} = \dfrac{4-1}{3} = 1$

(*ii*) $\quad \dfrac{4}{3} = \dfrac{\sqrt{y+6} - 1}{3} \Rightarrow 5 = \sqrt{y+6}$

$$\Rightarrow \quad y + 6 = 25 \Rightarrow y = 19$$

Or

Question asked in the examination is not correct.

Binary operation: $a * b = a - b + ab$ on set $A = Q - \{1\}$

Let $a = 0, b = -1$

then $0 * (-1) = 0 - (-1) + 0 \times (-1) = 0 + 1 - 0 = 1$

$\therefore$ $0 * (-1) = 1 \notin A$

Hence, operation is not a binary operation.

Otherwise

For Commutative: $a * b = a - b + ab, a, b \in A$

$b * a = b - a + ab, a, b \in A$

$a * b \neq b * a$, Hence, not commutative

For Associative: Let $a, b, c \in A$

$$\begin{aligned} a * (b * c) &= a * (b - c + bc) \\ &= a - (b - c + bc) + a(b - c + bc) \\ &= a - b + c - bc + ab - ac + abc \quad \text{...}(i) \end{aligned}$$

and
$$\begin{aligned} (a * b) * c &= (a - b + ab) * c \\ &= (a - b + ab) - c + (a - b + ab)c \\ &= a - b + ab - c + ac - bc + abc \quad \text{...}(ii) \end{aligned}$$

From (*i*) and (*ii*) we notice $(a * b) * c \neq a * (b * c)$.

Hence, Not associative.

For identity element: Let e be identity element

then $a * e = e * a = a$, for all $a \in A$

$a - e + ae = e - a + ae = a$

$a - e + ae = a \Rightarrow e(1 - a) = 0 \Rightarrow e = 0$ as $a \neq 1$

for $e - a + ae = a \Rightarrow e(1 + a) = 2a \Rightarrow e = \dfrac{2a}{1 + a}$

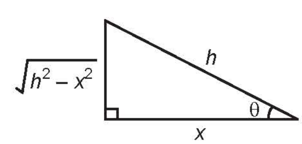
Not unique. No identity. Hence, no inverse.

26. If the sum of lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$.

Sol. Let side of the triangle be x and hypotenuse be h

$$x + h = G \text{ (given)} \quad \text{...}(i)$$

Area of the right-angled triangle, $A = \frac{1}{2}x \cdot \sqrt{h^2 - x^2}$



$$A = \frac{1}{2}x\sqrt{(G - x)^2 - x^2}$$

$$= \frac{1}{2}x\sqrt{G^2 - 2Gx}$$

If A is maximum, then A^2 is maximum

$$A^2 = B(\text{say}) = \frac{1}{4}x^2(G^2 - 2Gx) = \frac{1}{4}(G^2x^2 - 2Gx^3)$$

$$\frac{dB}{dx} = \frac{1}{4}(2xG^2 - 6Gx^2)$$

For maximum area, $\frac{dB}{dx} = 0 \Rightarrow 2xG^2 = 6Gx^2 \Rightarrow G = 3x$

$$\frac{d^2B}{dx^2} = \frac{1}{4}(2G^2 - 12Gx) \Rightarrow \frac{d^2B}{dx^2}\Big]_{G=3x} < 0$$

$\Rightarrow$ area is maximum for $G = 3x$

$\Rightarrow \quad x + h = 3x \Rightarrow h = 2x$ [from (i)]

$\Rightarrow \quad \frac{x}{h} = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

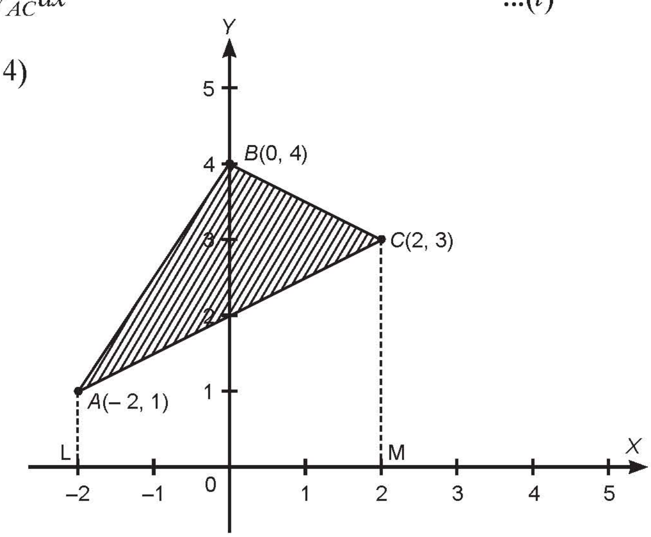
27. Using integration, find the area of region bounded by the triangle whose vertices are (–2, 1), (0, 4) and (2, 3).

Or

Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $\sqrt{3}y = x$ in the first quadrant, using integration.

Sol. Vertices of a triangle are $A(-2, 1)$, $B(0, 4)$ and $C(2, 3)$

$$\text{ar}(ABC) = \int_{-2}^{0} y_{AB}\,dx + \int_{0}^{2} y_{BC}\,dx - \int_{-2}^{2} y_{AC}\,dx \quad ...(i)$$



For equation of AB: $A(-2, 1)$ and $B(0, 4)$

$$m = \frac{4-1}{0+2} = \frac{3}{2}$$

$$\therefore \; y - 4 = \frac{3}{2}(x - 0) \Rightarrow y = \frac{3}{2}x + 4$$

For equation of BC: $B(0, 4)$ and $C(2, 3)$

$$m = \frac{3-4}{2-0} = -\frac{1}{2}$$

$$\therefore \; y - 4 = -\frac{1}{2}(x - 0) \Rightarrow y = -\frac{1}{2}x + 4$$

For equation of AC: $A(-2, 1)$ and $C(2, 3)$

$$m = \frac{3-1}{2+2} = \frac{2}{4} = \frac{1}{2}$$

$\therefore \quad y - 1 = \frac{1}{2}(x + 2) \Rightarrow y = \frac{x}{2} + 2$

From (*i*), $\quad \text{ar}(ABC) = \int_{-2}^{0}\left(\frac{3}{2}x + 4\right)dx + \int_{0}^{2}\left(-\frac{1}{2}x + 4\right)dx - \int_{-2}^{2}\left(\frac{1}{2}x + 2\right)dx$

$$= \left(\frac{3}{4}x^2 + 4x\right)_{-2}^{0} + \left(-\frac{x^2}{4} + 4x\right)_{0}^{2} - \left(\frac{x^2}{4} + 2x\right)_{-2}^{2}$$

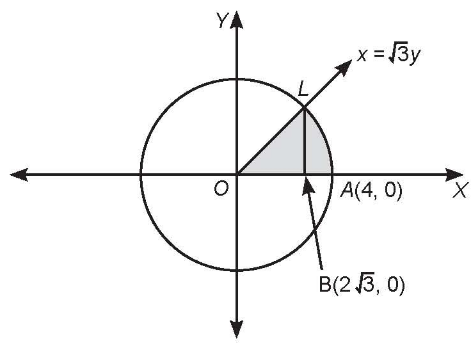
$$= (0) - (3 - 8) + (-1 + 8) - (0) - (1 + 4) + (1 - 4)$$

$$= 5 + 7 - 5 - 3 = 4 \text{ sq units}$$

Or

Given: circle $x^2 + y^2 = 16$ and line $x = \sqrt{3}y$

On plotting the curves, we find the shaded area as shown



Eliminating y from two equations, we get

$$x^2 + \frac{x^2}{3} = 16$$

$$\Rightarrow \quad \frac{4x^2}{3} = 16 \Rightarrow x^2 = 12$$

$$\Rightarrow \quad x = \pm 2\sqrt{3} \Rightarrow x = 2\sqrt{3} \text{ (first quadrant) and } y = 2$$

$\therefore$ Required area = area (OBL) + area (LBA)

$$= \int_{0}^{2\sqrt{3}} y_1\, dx + \int_{2\sqrt{3}}^{4} y_2\, dx$$

$$= \int_{0}^{2\sqrt{3}} \frac{x}{\sqrt{3}}dx + \int_{2\sqrt{3}}^{4} \sqrt{16 - x^2}\, dx \qquad \begin{pmatrix} y_1 : x = \sqrt{3}y \\ y_2 : x^2 + y^2 = 16 \end{pmatrix}$$

$$= \left[\frac{x^2}{2\sqrt{3}}\right]_0^{2\sqrt{3}} + \left\{\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right\}_{2\sqrt{3}}^{4}$$

$$= 2\sqrt{3} - 0 + \{0 + 8\sin^{-1}1\} - \left\{\frac{2\sqrt{3}}{2}\sqrt{16 - 12} + 8\sin^{-1}\frac{\sqrt{3}}{2}\right\}$$

$$= 2\sqrt{3} + 8.\frac{\pi}{2} - 2\sqrt{3} - 8.\frac{\pi}{3} = \frac{4\pi}{3} \text{ sq units.}$$

28. Solve the differential equation $x\frac{dy}{dx} + y = x\cos x + \sin x$, given that $y = 1$ when $x = \frac{\pi}{2}$.

Sol. Consider equation $x\frac{dy}{dx} + y = x\cos x + \sin x$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}$$

Here, $P(x) = \frac{1}{x}, Q(x) = \cos x + \frac{\sin x}{x}$

$$\text{Integrating factor} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$\text{Solution is (I.F.)}\, y = \int \{(\text{I.F.})\, Q(x)\}\, dx$$

$$x.y = \int x.\left(\cos x + \frac{\sin x}{x}\right)dx = \int x \cos x\, dx + \int \sin x\, dx$$

$$= x \sin x - \int 1 . \sin x\, dx + \int \sin x\, dx$$

$$\Rightarrow \quad xy = x \sin x + C \qquad ...(i)$$

Given $y = 1$, when $x = \frac{\pi}{2}$

$$\Rightarrow \quad \frac{\pi}{2}.1 = \frac{\pi}{2}.\sin\frac{\pi}{2} + C \quad \Rightarrow \quad C = 0$$

$\therefore$ From (*i*), solution is $xy = x \sin x \Rightarrow y = \sin x.$

29. Find the equation of the plane through the line of intersection of $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $\vec{r} \cdot (\hat{i} - \hat{j}) + 4 = 0$ and perpendicular to the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$. Hence find whether the plane thus obtained contains the line $x - 1 = 2y - 4 = 3z - 12$.

Or

Find the vector and Cartesian equations of a line passing through (1, 2, – 4) and perpendicular to the two lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

Sol. General equation of a plane passing through intersection of plane $\vec{r} . (2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $\vec{r} . (\hat{i} - \hat{j}) + 4 = 0$ is $\vec{r} . \{(2\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda(\hat{i} - \hat{j})\} - 1 + 4\lambda = 0$

i.e. $\vec{r} . \{(2 + \lambda)i + (-3 - \lambda)\hat{j} + 4\hat{k}\} - 1 + 4\lambda = 0$ ...(*i*)

If plane (*i*) is perpendicular to the plane $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$

then $2(2 + \lambda) - 1(-3 - \lambda) + 1 \times 4 = 0$

$$\Rightarrow \quad 4 + 2\lambda + 3 + \lambda + 4 = 0 \quad \Rightarrow \quad 3\lambda + 11 = 0 \quad \Rightarrow \quad \lambda = -\frac{11}{3}$$

Substituting in (*i*), we get

$$\vec{r} . \left\{\left(2 - \frac{11}{3}\right)\hat{i} + \left(-3 + \frac{11}{3}\right)\hat{j} + 4\hat{k}\right\} - 1 - \frac{44}{3} = 0$$

$$\Rightarrow \quad \vec{r} . \left\{-\frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + 4\hat{k}\right\} - \frac{47}{3} = 0$$

$\Rightarrow \quad \vec{r}\cdot(5\hat{i} - 2\hat{j} - 12\hat{k}) + 47 = 0$ is equation of the plane.

In Cartesian form equation is $5x - 2y - 12z + 47 = 0$ ...(*ii*)

Given line is $x - 1 = 2y - 4 = 3z - 12$

$$\Rightarrow \quad 1(x-1) = 2(y-2) = 3(z-4)$$

$$\Rightarrow \quad \frac{x-1}{6} = \frac{y-2}{3} = \frac{z-4}{2}$$

If line contains plane (*ii*), then point (1, 2, 4) on line must lie in plane

i.e. $5 - 4 - 48 + 47 = 0 \quad \Rightarrow \quad 52 - 52 = 0$, lies

and $5 \times 6 + (-2) \times 3 + (-12) \times 2 = 0 \quad \Rightarrow \quad 30 - 6 - 24 = 0$, true

Hence, line lies in the plane.

Or

Equation of line in vector form are

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $$\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$$

Let line through the point (1, 2, –4) be

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda'\vec{m}, \text{ where } \lambda' \text{ is a scalar} \quad ...(i)$$

Line (*i*) is perpendicular to lines

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $$\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$$

$$\therefore \quad (3\hat{i} - 16\hat{j} + 7\hat{k})\cdot\vec{m} = 0 \text{ and } (3\hat{i} + 8\hat{j} - 5\hat{k})\cdot\vec{m} = 0$$

$$\Rightarrow \quad \vec{m} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} = 24\hat{i} + 36\hat{j} + 72\hat{k}$$

$\therefore$ From (*i*) line is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda'(24\hat{i} + 36\hat{j} + 72\hat{k})$$

or $$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda''(2\hat{i} + 3\hat{j} + 6\hat{k}), \text{ where } \lambda'' = 12\lambda', \text{ is a scalar}$$

Vector form of line is $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda''(2\hat{i} + 3\hat{j} + 6\hat{k})$

$\therefore$ Point through which line passes is (1, 2, – 4) and DR's of line are 2, 3, 6

$\therefore$ Cartesian form is $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z+4}{6}$.

[SET II UNCOMMON QUESTIONS TO SET I]

SECTION – B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

12. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then find the rate of change of the slope of the curve when $x = 3$.

Sol. Given $\dfrac{dx}{dt} = 2$ units/sec.

Equation of curve is $y = 5x - 2x^3 \Rightarrow \dfrac{dy}{dx} = 5 - 6x^2$

$\therefore$ Slope of the curve, $m = 5 - 6x^2$

$$\frac{dm}{dt} = 0 - 12x \cdot \frac{dx}{dt} \Rightarrow \frac{dm}{dt} = -12x \times 2 = -24x$$

$$\therefore \quad \left.\frac{dm}{dt}\right]_{x=3} = -24 \times 3 = -72$$

Slope decreases at the rate of 72 units/sec.

SECTION – C

Question numbers **13** *to* **23** *carry* **4** *marks each.*

20. The random variable X can take only the values 0, 1, 2, 3. Given that $P(2) = P(3) = p$ and $P(0) = 2P(1)$. If $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$, find the value of p.

Sol. X can take values 0, 1, 2, 3.

Given $P(2) = P(3) = p$ and $P(0) = 2P(1) = x$ (say)

$$\therefore \quad P(0) + P(1) + P(2) + P(3) = 1$$

$$\Rightarrow \quad x + \frac{x}{2} + p + p = 1 \Rightarrow \frac{3x}{2} = 1 - 2p \Rightarrow x = \frac{2}{3}(1 - 2p)$$

Distribution is

X	$P(X)$	$X \cdot P(X)$	$X^2 \cdot P(X)$
0	$\frac{2}{3}(1-2p)$	0	0
1	$\frac{1}{3}(1-2p)$	$\frac{1}{3}(1-2p)$	$\frac{1}{3}(1-2p)$
2	p	$2p$	$4p$
3	p	$3p$	$9p$

Also $\qquad \Sigma p_i x_i^2 = 2\Sigma p_i x_i$

$\Rightarrow \qquad \frac{1}{3}(1-2p) + 4p + 9p = 2[\frac{1}{3}(1-2p) + 2p + 3p]$

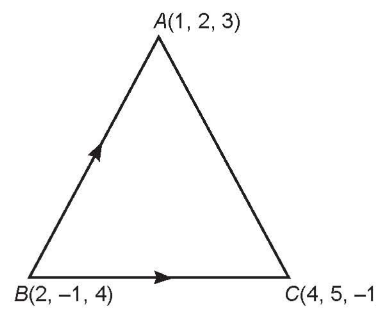
$\Rightarrow \qquad \frac{1}{3} - \frac{2}{3}p + 13p = \frac{2}{3} - \frac{4}{3}p + 10p$

$\Rightarrow \qquad \frac{4}{3}p - 10p - \frac{2}{3}p + 13p = \frac{2}{3} - \frac{1}{3}$

$\Rightarrow \qquad \frac{2}{3}p + 3p = \frac{1}{3} \Rightarrow \frac{11}{3}p = \frac{1}{3} \Rightarrow p = \frac{1}{11}$

21. Using vectors find the area of triangle ABC with vertices $A(1, 2, 3)$, $B(2, -1, 4)$, and $C(4, 5, -1)$.

Sol. Area of triangle $= \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}|$



$$\overrightarrow{BC} = (4-2)\hat{i} + (5+1)\hat{j} + (-1-4)\hat{k}$$
$$= 2\hat{i} + 6\hat{j} - 5\hat{k}$$
$$\overrightarrow{BA} = (1-2)\hat{i} + (2+1)\hat{j} + (3-4)\hat{k}$$
$$= -\hat{i} + 3\hat{j} - \hat{k}$$

$$\overrightarrow{BC} \times \overrightarrow{BA} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & -5 \\ -1 & 3 & -1 \end{vmatrix} = \hat{i}(-6+15) - \hat{j}(-2-5) + \hat{k}(6+6) = 9\hat{i} + 7\hat{j} + 12\hat{k}$$

$$|\overrightarrow{BC} \times \overrightarrow{BA}| = \sqrt{81+49+144} = \sqrt{274}$$

Area of triangle $= \frac{1}{2}\sqrt{274}$ sq units

22. Solve the following L.P.P. graphically

Maximise $\qquad Z = 4x + y$

Subject to following constraints $\qquad x + y \leq 50,$

$3x + y \leq 90,$

$x \geq 10$

$x, y \geq 0$

Sol. To maximise $\qquad Z = 4x + y$

Subject to constraints

$x, y \geq 0, x \geq 10$

$x + y \leq 50$

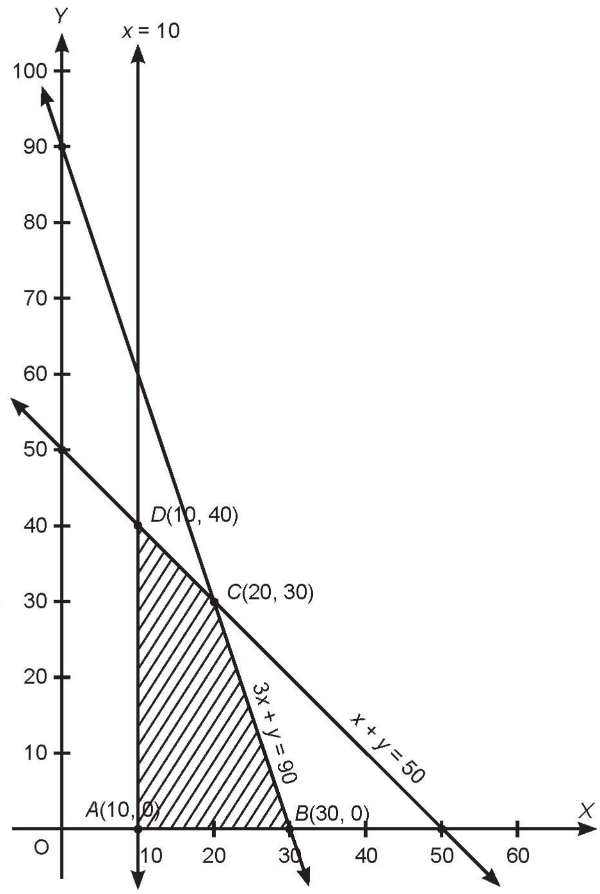
$3x + y \leq 90$

Plotting the inequations, we notice shaded portion is feasible solution. Possible points for maximum Z are $A(10, 0)$, $B(30, 0)$, $C(20, 30)$ and $D(10, 40)$

Points	$Z = 4x + y$	Value	
$A(10, 0)$	$40 + 0$	40	
$B(30, 0)$	$120 + 0$	120	←Maximum
$C(20, 30)$	$80 + 30$	110	
$D(10, 40)$	$40 + 40$	80	

Z is maximum at $B(30, 0)$, i.e. $x = 30$, $y = 0$.

$\therefore$ For $x = 30, y = 0$, Z is maximum.



23. **Find** $\int \frac{2x}{(x^2+1)(x^4+4)}dx$

Sol. Consider $\int \frac{2x}{(x^2+1)(x^4+4)}dx$

Let $x^2 = t$
$\Rightarrow$ $2x\,dx = dt$

$$= \int \frac{1}{(t+1)(t^2+4)}dt$$

Let $$\frac{1}{(t+1)(t^2+4)} \equiv \frac{A}{t+1} + \frac{Bt+C}{t^2+4} \quad ...(i)$$

$\Rightarrow$ $$1 = A(t^2 + 4) + (Bt + C)(t + 1)$$
$$= At^2 + 4A + Bt^2 + Bt + Ct + C$$
$$1 = t^2(A + B) + t(B + C) + (4A + C)$$

Now, comparing the coefficients of t^2, t and constant terms on both sides, we get

$$A + B = 0 \Rightarrow A = -B$$

$$B + C = 0 \Rightarrow B = -C \Rightarrow A = C$$

$$4A + C = 1 \Rightarrow A = C = \frac{1}{5} \text{ and } B = -\frac{1}{5}$$

Substituting in (*i*) and integrating, we get

$$\int \frac{1}{(t+1)(t^2+1)}dt = \frac{1}{5}\int \frac{1}{t+1}dt + \frac{1}{5}\int \frac{-t+1}{t^2+4}dt$$

$$= \frac{1}{5}\int \frac{1}{t+1}dt - \frac{1}{10}\int \frac{2t}{t^2+4}dt + \frac{1}{5}\int \frac{1}{t^2+4}dt$$

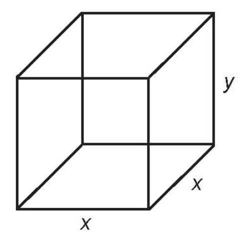
$$= \frac{1}{5}\log|t+1| - \frac{1}{10}\log|t^2+4| + \frac{1}{10}\tan^{-1}\frac{t}{2} + C$$

$$\therefore \quad \int \frac{2x}{(x^2+1)(x^4+4)}dx = \frac{1}{5}\log|x^2+1| - \frac{1}{10}\log|x^4+4| + \frac{1}{10}\tan^{-1}\frac{x^2}{2} + C$$

SECTION – D

Question numbers **24** *to* **29** *carry* **6** *marks each.*

28. A metal box with a square base and vertical sides is to contain 1024 cm^3. The material for the top and bottom costs ₹ 5 per cm^2 and the material for the sides costs ₹ 2.50 per cm^2. Find the least cost of the box.



Sol. Let bases sides be x each and vertical sides be y each.

$$\therefore \quad x \cdot x \cdot y = 1024 \Rightarrow x^2y = 1024 \quad \text{...}(i)$$

Also

$$\text{cost}(C) = 5 \times 2x^2 + \frac{5}{2} \times 2(x + x) \times y$$

$$= 10x^2 + 10xy$$

$$\Rightarrow \quad C = 10x^2 + 10x \times \frac{1024}{x^2} \quad \text{[From } (i)\text{]}$$

$$\Rightarrow \quad C = 10x^2 + \frac{10240}{x} \quad \text{...}(ii)$$

$$\frac{dC}{dx} = 20x - \frac{10240}{x^2}$$

For minimum cost, $\frac{dC}{dx} = 0$

$$\Rightarrow \quad 20x = \frac{10240}{x^2} \Rightarrow x^3 = 512 = (8)^3 \Rightarrow x = 8$$

$$\frac{d^2C}{dx^2} = 20 + \frac{20480}{x^3}$$

$$\left.\frac{d^2C}{dx^2}\right|_{x=8} = 20 + \frac{20480}{(8)^3} > 0$$

$\therefore$ For $x = 8$, cost is minimum.

Substituting in (*ii*), we get

minimum cost of the box $= 10(8)^2 + \dfrac{10240}{8} = 640 + 1280 =$ ₹ 1920

29. If $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$**, find** A^{-1}**. Using** A^{-1} **solve the system of equations**

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 2; \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 5; \quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = -4$$

Sol. Consider $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$

We have $A^{-1} = \dfrac{1}{|A|}(\text{Adj. } A)$

$$|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$$

$$= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$$

$$= 150 + 330 + 720 = 1200 \neq 0$$

Hence, A^{-1} exists.

Matrix formed by cofactors of each element in $|A|$.

$$\begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}$$

$$\therefore \quad \text{Adj. } A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}' = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Consider equations

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 2$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 5$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = -4$$

Corresponding matrix equation is

$$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$AX = B$ is matrix equation.

Its solution is $X = A^{-1}B$

$$\Rightarrow \qquad X = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 150+750-300 \\ 220-500-120 \\ 144+0+96 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ -400 \\ 240 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\Rightarrow \qquad \frac{1}{x} = \frac{1}{2}, \frac{1}{y} = -\frac{1}{3}, \frac{1}{z} = \frac{1}{5}$$

$$\Rightarrow \qquad x = 2, y = -3, z = 5.$$

[SET III UNCOMMON QUESTIONS TO SET I AND SET II]

SECTION – B

Question numbers **5** *to* **12** *carry* **2** *marks each.*

12. If $y = \sin^{-1}(6x\sqrt{1-9x^2}), -\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$, then find $\frac{dy}{dx}$.

Sol. Consider $\quad y = \sin^{-1}\left(6x\sqrt{1-9x^2}\right)$

$$= \sin^{-1}\left(2.(3x)\sqrt{1-(3x)^2}\right)$$

Let $3x = \sin\theta$
$\Rightarrow \quad \theta = \sin^{-1}(3x)$

$$= \sin^{-1}\left(2\sin\theta \,.\, \sqrt{1-\sin^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$y = 2\sin^{-1}(3x)$$

$$\therefore \quad \frac{dy}{dx} = 2\,.\,\frac{1}{\sqrt{1-(3x)^2}}\,.\,3 = \frac{6}{\sqrt{1-9x^2}}.$$

SECTION – C

Question numbers **13** *to* **23** *carry* **4** *marks each*.

20. Solve the following L.P.P. graphically:

Maximise $\quad Z = 20x + 10y$

Subject to the following constraints $\quad x + 2y \le 28,$

$$3x + y \le 24,$$

$$x \ge 2,$$

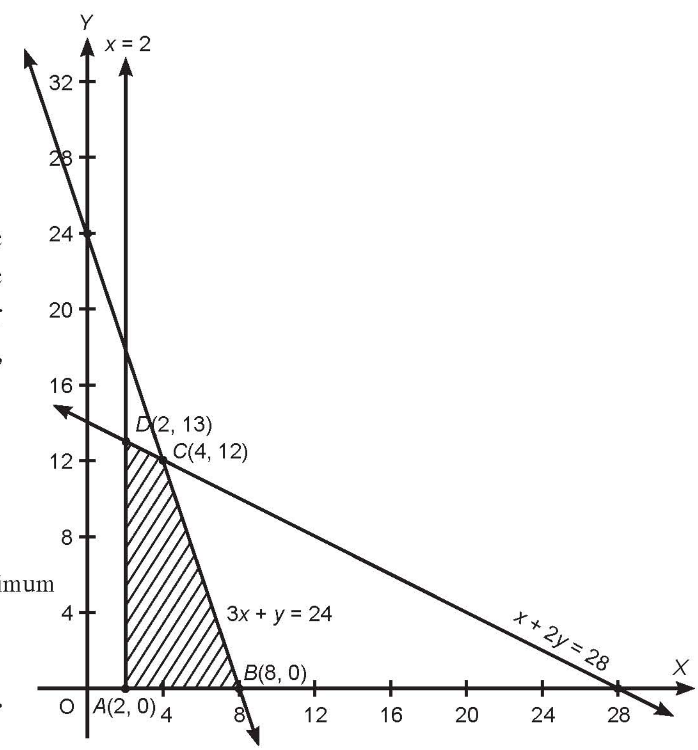
$$x, y \ge 0$$

Sol. To maximise $Z = 20x + 10y$

Subject to the constraints

$$x, y \ge 0$$
$$x \ge 2$$
$$x + 2y \le 28$$
$$3x + y \le 24$$

On plotting the inequations, we notice shaded portion is feasible solution. Possible points for maximum Z are $A(2, 0)$, $B(8, 0)$, $C(4, 12)$ and $D(2, 13)$



Points	$Z = 20x + 10y$	Value	
$A(2, 0)$	40 + 0	40	
$B(8, 0)$	160 + 0	160	
$C(4, 12)$	80 + 120	200	←Maximum
$D(2, 13)$	40 + 130	170	

Z is maximum at $C(4, 12)$, i.e. $x = 4, y = 12$.

Hence, for $x = 4$ and $y = 12$, Z is maximum. Maximum value is 200.

21. Show that the family of curves for which $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

Sol. Consider $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ ...(*i*)

Let $y = vx$

$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (*i*), we get

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2vx^2}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\Rightarrow \quad \frac{2v}{1-v^2}dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{2v}{1-v^2}dv = \int \frac{dx}{x}$$

$$-\int \frac{1}{t}dt = \int \frac{dx}{x}$$

| Let $1 - v^2 = t$
| $\Rightarrow \quad -2v\,dv = dt$

$$\Rightarrow \quad -\log|t| + \log|c| = \log|x|$$

$$\Rightarrow \quad \log\left|\frac{c}{t}\right| = \log|x| \quad \Rightarrow \quad c = xt$$

$$\Rightarrow \quad c = x(1-v^2) = x\left(1-\frac{y^2}{x^2}\right) \quad \Rightarrow \quad c = \frac{x^2-y^2}{x}$$

$$\Rightarrow \quad x^2 - y^2 = cx$$

22. Find $\int \frac{(3\sin x - 2)\cos x}{13 - \cos^2 x - 7\sin x}dx$

Sol. Consider $\int \frac{(3\sin x - 2)\cos x}{13 - \cos^2 x - 7\sin x}dx$

$$= \int \frac{(3\sin x - 2)\cos x}{12 + \sin^2 x - 7\sin x}dx \qquad \left| \begin{array}{l} \text{Let} \quad \sin x = t \\ \Rightarrow \quad \cos x\, dx = dt \end{array} \right.$$

$$= \int \frac{3t - 2}{t^2 - 7t + 12}dt = \int \frac{3t-2}{(t-3)(t-4)}dt$$

Let $\quad \dfrac{3t-2}{(t-3)(t-4)} = \dfrac{A}{t-3} + \dfrac{B}{t-4}$ ...(*i*)

$\Rightarrow \quad 3t - 2 = A(t-4) + B(t-3)$

$\Rightarrow \quad 3t - 2 = t(A + B) + (-4A - 3B)$

Comparing the coefficients of t and constant terms on both sides, we get

$$A + B = 3$$

$$-4A - 3B = -2 \Rightarrow A = -7, B = 10$$

Substituting in (*i*), and integrating, we get

$$\int \frac{3t-2}{(t-3)(t-4)}dt = \int \frac{-7}{t-3}dt + \int \frac{10}{t-4}dt$$

$$= -7 \log |t - 3| + 10 \log |t - 4| + C$$

$$\therefore \int \frac{(3\sin x - 2)\cos x}{13 - \cos^2 x - 7\sin x}dx = -7\log|\sin x - 3| + 10\log|\sin x - 4| + C$$

23. Solve the following equation for x: $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$.

Sol. Consider the equation,

$$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$

$$= \cos\left[\frac{\pi}{2} - \cot^{-1}\frac{3}{4}\right] \qquad \left[\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)\right]$$

$$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{2} - \cot^{-1}\frac{3}{4} = \tan^{-1}\frac{3}{4} \qquad \left[\tan^{-1}\theta = \frac{\pi}{2} - \cot^{-1}\theta\right]$$

$$\Rightarrow \quad x = \frac{3}{4}$$

Alternate method:

Consider equation,

$$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$

$$\Rightarrow \quad \frac{1}{\sec(\tan^{-1}x)} = \frac{1}{\operatorname{cosec}\left(\cot^{-1}\frac{3}{4}\right)}$$

$$\Rightarrow \qquad \sec(\tan^{-1}x) = \operatorname{cosec}\left(\cot^{-1}\frac{3}{4}\right)$$

$$\Rightarrow \qquad \sqrt{1+\tan^2(\tan^{-1}x)} = \sqrt{1+\cot^2\left(\cot^{-1}\frac{3}{4}\right)}$$

$$\Rightarrow \qquad 1 + x^2 = 1 + \frac{9}{16}$$

$$\Rightarrow x^2 = \frac{9}{16} \quad \Rightarrow \quad x = \pm\frac{3}{4}$$

SECTION – D

Question numbers **24** *to* **29** *carry* **6** *marks each***.**

28. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ -3 & 1 & -1 \end{bmatrix}$**, find** A^{-1} **and hence solve the system of equations** $2x + y - 3z = 13$, $3x + 2y + z = 4, x + 2y - z = 8$.

Sol. Consider, $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ -3 & 1 & -1 \end{bmatrix}$ ...(*i*)

We have $A^{-1} = \dfrac{1}{|A|}\text{Adj}A$

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ -3 & 1 & -1 \end{vmatrix}$$

$$= 2(-2-2) - 3(-1+6) + 1(1+6)$$

$$= -8 - 15 + 7 = -16 \neq 0$$

Hence, A^{-1} exists.

Matrix formed by cofactors of each element in $|A|$ is $\begin{bmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{bmatrix}$

$$\therefore \qquad \text{Adj}A = \begin{bmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{bmatrix}' = \begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix}$$

$$\therefore \qquad A^{-1} = -\frac{1}{16}\begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix} \qquad ...(ii)$$

Consider equations

$$2x + y - 3z = 13$$
$$3x + 2y + z = 4$$
$$x + 2y - z = 8$$

Matrix equation is

$$\begin{bmatrix} 2 & 1 & -3 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 4 \\ 8 \end{bmatrix}$$

$\Rightarrow \quad A^T.X = B$ [From (i)]

$\Rightarrow \quad X = (A^T)^{-1}B$ is its solution

$\Rightarrow \quad X = (A^{-1})^{T}B$

$$= \left[-\frac{1}{16}\begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix}\right]'\begin{bmatrix} 13 \\ 4 \\ 8 \end{bmatrix}$$

$$= -\frac{1}{16}\begin{bmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 13 \\ 4 \\ 8 \end{bmatrix} \qquad [\because (A^T)^{-1} = (A^{-1})^T]$$

$$\Rightarrow \quad X = -\frac{1}{16}\begin{bmatrix} -52 - 20 + 56 \\ 52 + 4 - 88 \\ 52 - 12 + 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{16}\begin{bmatrix} -16 \\ -32 \\ 48 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$\Rightarrow x = 1, y = 2, z = -3$ is solution

29. Find the particular solution of the differential equation

$\tan x.\dfrac{dy}{dx} = 2x\tan x + x^2 - y$; ($\tan x \neq 0$) given that $y = 0$ when $x = \dfrac{\pi}{2}$.

Sol. Consider equation $\tan x.\dfrac{dy}{dx} = 2x\tan x + x^2 - y$

$$\Rightarrow \quad \tan x.\frac{dy}{dx} + y = 2x\tan x + x^2$$

$$\Rightarrow \quad \frac{dy}{dx} + \cot x \cdot y = (2x\tan x + x^2)\cot x$$

Here $P(x) = \cot x$, $Q(x) = (2x\tan x + x^2)\cot x$

Integrating factor ($I.F.$) $= e^{\int \cot x\, dx} = e^{\log|\sin x|} = \sin x$

$\therefore$ Solution is $(I.F.)y = \int (I.F.)Q(x)dx$

$$\Rightarrow \quad \sin x.y = \int \sin x(2x \tan x + x^2).\cot x \, dx$$

$$= \int (2x \sin x + x^2 \cos x) dx$$

$$= \int 2x \sin x \, dx + \int \underset{①}{x^2} \underset{②}{\cos x} \, dx$$

$$= \int 2x \sin x \, dx + x^2 \cdot \sin x - \int 2x \cdot \sin x \, dx$$

$$\sin x.y = x^2 \sin x + C \qquad ...(i)$$

Given $y = 0$, when $x = \frac{\pi}{2}$

$$0 = \frac{\pi^2}{4} \cdot \sin\frac{\pi}{2} + C \quad \Rightarrow \quad C = -\frac{\pi^2}{4}$$

Substituting in (i), we get

$$\sin x.y = x^2 \sin x - \frac{\pi^2}{4}$$ is required solution.

Examination Papers, 2017
[All India Set-I, II, III]

1. If for any 2 × 2 square matrix A, $A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, then write the value of $|A|$.

Sol. $A.(\text{adj } A) = |A| I \Rightarrow A(\text{adj } A) = 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 8I \Rightarrow |A| = 8$

2. Determine the value of 'k' for which the following function is continuous at $x = 3$:

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3} & , \; x \neq 3 \\ k & , \; x = 3 \end{cases}$$

Sol. If continuous at $x = 3$, then $\lim_{x \to 3} f(x) = f(3)$

$$\Rightarrow \quad \lim_{x \to 3} \frac{(x+3)^2 - 36}{x-3} = k$$

$$\Rightarrow \quad \lim_{t \to 0} \frac{(t+6)^2 - 36}{t} = k \qquad \left| \begin{array}{l} \text{Let } x - 3 = t \\ \text{as } x \to 3, t = 0 \end{array} \right.$$

$$\Rightarrow \quad \lim_{t \to 0} \frac{t^2 + 12t + 36 - 36}{t} = k$$

$$\Rightarrow \quad \lim_{t \to 0} (t + 12) = k$$

$$\Rightarrow \quad 12 = k \Rightarrow k = 12$$

3. Find: $\int \dfrac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$

Sol. $\int \dfrac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx = \int (\tan x - \cot x) dx = \log|\sec x| - \log|\sin x| + C$

4. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$.

Sol. Planes are $\quad 2x - y + 2z = 5 \quad \Rightarrow \quad 2x - y + 2z - 5 = 0$

and $\quad 5x - 2.5y + 5z = 20 \quad \Rightarrow \quad 2x - y + 2z - 8 = 0$

$$\text{Distance} = \left| \frac{-5+8}{\sqrt{4+1+4}} \right| = \left| \frac{3}{3} \right| = 1 \text{ unit}$$

5. If A is a skew-symmetric matrix of order 3, then prove that $\det A = 0$.

Sol. Let $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$, skew symmetric matrix of order 3

$$|A| = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = \frac{1}{ab}\begin{vmatrix} 0 & ab & ab \\ -a & 0 & ac \\ -b & -bc & 0 \end{vmatrix}$$

[Performing $C_2 \to bC_2, C_3 \to aC_3$ and dividing determinant by ab]

$$= \frac{1}{ab}\begin{vmatrix} 0 & 0 & ab \\ -a & -ac & ac \\ -b & -bc & 0 \end{vmatrix}$$ [Performing $C_2 \to C_2 - C_3$]

$$= \frac{1}{ab}[ab(abc - abc)] = 0$$

6. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$.

Sol. $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$

(*i*) f is continuous in $[-\sqrt{3}, 0]$ as polynomial function is always continuous.

(*ii*) $f'(x) = 3x^2 - 3$, exists in $[-\sqrt{3}, 0]$, hence f is derivable

(*iii*) $f(-\sqrt{3}) = -3\sqrt{3} + 3\sqrt{3} = 0$,

$f(0) = 0, \;\therefore\; f(-\sqrt{3}) = f(0)$

$\therefore$ Conditions of Rolle's Theorem are satisfied.

Hence, there exists at least one point $c \in (-\sqrt{3}, 0)$ such that

$$f'(c) = 0 \Rightarrow 3c^2 - 3 = 0 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1$$

$$\therefore \quad c = -1 \in (-\sqrt{3}, 0)$$

7. The volume of a cube is increasing at the rate of 9 cm^3/s. How fast is its surface area increasing when the length of an edge is 10 cm?

Sol. Let V be volume, S be surface area and x be side of the cube.

Volume $V = x^3$,

Surface area $= 6x^2$, where x is function of time t.

Given $$\frac{dV}{dt} = 9 \Rightarrow \frac{d}{dt}(x^3) = 9 \Rightarrow 3x^2.\frac{dx}{dt} = 9 \Rightarrow \frac{dx}{dt} = \frac{3}{x^2} \quad ...(i)$$

$$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = 12x.\frac{dx}{dt} = 12x.\frac{3}{x^2} = \frac{36}{x}$$

$$\therefore \quad \left.\frac{dS}{dt}\right]_{x=10} = \frac{36}{10} = 3.6 \text{ cm}^2\text{/s}$$

Hence, surface area is increasing at the rate of 3.6 cm^2/s when $x = 10$ cm.

8. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R.

Sol. Consider function

$$f(x) = x^3 - 3x^2 + 6x - 100$$

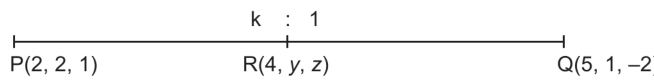
$$f'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 2)$$

$$= 3[(x-1)^2 + 1] > 0, \text{ for all } x \in R$$

Hence, f is increasing on R.

9. The x-coordinate of a point on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is 4. Find its z-coordinate.

Sol. Let point $R(4, y, z)$ lies on the line joining $P(2, 2, 1)$ and $Q(5, 1, -2)$. Let R divides PQ in ratio $k : 1$



$$\therefore \quad \frac{5k+2}{k+1} = 4 \Rightarrow 5k + 2 = 4k + 4 \Rightarrow k = 2$$

$$\therefore \quad z\text{-coordinate} = \frac{-2k+1}{k+1} = \frac{-4+1}{2+1} = -1$$

10. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events.

Sol. A : number is even, i.e. 2, 4, 6; $P(A) = \frac{3}{6} = \frac{1}{2}$

B : number is red, i.e. 1, 2, 3; $P(B) = \frac{3}{6} = \frac{1}{2}$

$A \cap B$: number is even and red, i.e. 2; $P(A \cap B) = \frac{1}{6}$

$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

As $P(A \cap B) \neq P(A) \cdot P(B)$

Hence, not independent.

11. Two tailors, A and B, earn ₹300 and ₹400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP.

Sol. Let A works for x days and B for y days

$\therefore$ LPP is to minimise $Z = 300x + 400y$

subject to the constraints

$$6x + 10y \geq 60$$

$$4x + 4y \geq 32$$

$$x \geq 0, \; y \geq 0$$

12. Find: $\displaystyle\int \frac{dx}{5-8x-x^2}$

Sol. $$\int \frac{dx}{5-8x-x^2} = \int \frac{dx}{5-(x^2+8x)} = \int \frac{dx}{5-\{(x+4)^2-16\}}$$

$$= \int \frac{dx}{21-(x+4)^2} = \frac{1}{2\sqrt{21}}\log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right| + C$$

13. If $\tan^{-1}\dfrac{x-3}{x-4} + \tan^{-1}\dfrac{x+3}{x+4} = \dfrac{\pi}{4}$**, then find the value of** x**.**

Sol. We have $\tan^{-1}\dfrac{x-3}{x-4} + \tan^{-1}\dfrac{x+3}{x+4} = \dfrac{\pi}{4}$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\dfrac{x-3}{x-4}+\dfrac{x+3}{x+4}}{1-\dfrac{x^2-9}{x^2-16}}\right] = \frac{\pi}{4} \qquad \left[\text{Considering } \frac{x^2-9}{x^2-16} < 1\right]$$

$$\Rightarrow \quad \frac{(x^2+x-12)+(x^2-x-12)}{x^2-16-x^2+9} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x^2-24}{-7} = 1 \Rightarrow 2x^2-24 = -7$$

$$\Rightarrow \quad 2x^2 = 17 \Rightarrow x = \pm\sqrt{\frac{17}{2}}$$

14. Using properties of determinants, prove that $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$

Or

Find matrix A **such that** $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$

Sol. $$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$ [by performing $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$]

$$= (a-1)^2 \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$ [by taking $(a-1)$ common from R_1 and R_2]

$$= (a-1)^2\,[0-0+1(a+1-2)] = (a-1)^3$$

Or

For product to define matrix A should be 2×2 matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2a-c & 2b-d \\ a & b \\ -3a+4c & -3b+4d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

$$\Rightarrow \quad 2a - c = -1, 2b - d = -8, a = 1, b = -2 \qquad ...(i)$$

$$-3a + 4c = 9, -3b + 4d = 22 \qquad ...(ii)$$

From (i) when $a = 1, c = 3$, when $b = -2, d = 4$

We notice these values satisfy (ii) also

$$\therefore \quad A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

15. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

Or

If $e^y(x + 1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Sol.

$$e^{y\log x} + e^{x\log y} = e^{b\log a}$$

$$\Rightarrow e^{y\log x}\left\{y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}\right\} + e^{x\log y}\left\{x\cdot\frac{1}{y}\cdot\frac{dy}{dx} + \log y\cdot 1\right\} = 0$$

$$\Rightarrow \quad x^y\left\{\frac{y}{x} + \log x\frac{dy}{dx}\right\} + y^x\left\{\frac{x}{y}\frac{dy}{dx} + \log y\right\} = 0$$

$$\Rightarrow yx^{y-1} + x^y \log x\frac{dy}{dx} + xy^{x-1}\frac{dy}{dx} + y^x\log y = 0$$

$$\Rightarrow \quad \frac{dy}{dx}[x^y\log x + xy^{x-1}] = -[y^x\log y + yx^{y-1}]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-[y^x\log y + yx^{y-1}]}{[x^y\log x + xy^{x-1}]}$$

Or

$$e^y = \frac{1}{1+x} \Rightarrow y = -\log(1+x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{1+x} \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{(1+x)^2} \Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2.$$

16. Find: $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$

Sol. Consider, $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta = \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}d\theta$

$$= \int \frac{1}{(4+t^2)(1+4t^2)}dt \qquad \left| \begin{array}{l}\text{Let } \sin\theta = t \\ \Rightarrow \cos\theta\, d\theta = dt\end{array}\right.$$

$$= \frac{1}{15}\int \frac{[4(4+t^2)-(1+4t^2)]}{(4+t^2)(1+4t^2)}dt$$

$$= \frac{1}{15}\left[\int \frac{4}{1+4t^2}dt - \int \frac{1}{4+t^2}dt\right]$$

$$= \frac{1}{15}\left[\int \frac{1}{\left(\frac{1}{2}\right)^2+t^2}dt - \int \frac{1}{(2)^2+t^2}dt\right]$$

$$= \frac{1}{15}\left[2\tan^{-1}2t - \frac{1}{2}\tan^{-1}\frac{t}{2}\right] + C$$

$$= \frac{2}{15}\tan^{-1}(2\sin\theta) - \frac{1}{2}\tan^{-1}\left(\frac{\sin\theta}{2}\right) + C$$

17. Evaluate: $\int_0^\pi \frac{x\tan x}{\sec x+\tan x}dx$

Or

Evaluate: $\int_1^4 \{|x-1|+|x-2|+|x-4|\}\,dx$

Sol. Let $I = \int_0^\pi \frac{x\tan x}{\sec x+\tan x}dx = \int_0^\pi \frac{x\sin x}{1+\sin x}dx$ ...(*i*)

Using property $\int_0^a f(x)dx = \int_0^a f(a-x)\,dx$, we get

$$I = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}\,dx = \int_0^\pi \frac{\pi\sin x - x\sin x}{1+\sin x} \quad ...(ii)$$

$\therefore$ $$2I = \pi\int_0^\pi \frac{\sin x}{1+\sin x}\,dx \qquad \text{[Adding } (i) \text{ and } (ii)]$$

$$= \pi\int_0^\pi \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$$

$$= \pi\int_0^{\pi}(\sec x \tan x - \tan^2 x)\,dx$$

$$= \pi\int_0^{\pi}(\sec x \tan x - \sec^2 x + 1)\,dx$$

$$= \pi\Big[\sec x - \tan x + x\Big]_0^{\pi}$$

$$= \pi[(\sec \pi - \tan \pi + \pi) - (\sec 0 - \tan 0 + 0)]$$

$$2I = \pi(-2 + \pi)$$

$$\Rightarrow \qquad I = \frac{\pi}{2}(\pi - 2) = \pi\left(\frac{\pi}{2} - 1\right)$$

Or

	1 to 2	2 to 4
$\lvert x-1\rvert$:	$(x-1)$	$(x-1)$
$\lvert x-2\rvert$:	$-(x-2)$	$(x-2)$
$\lvert x-4\rvert$	$-(x-4)$	$-(x-4)$

$$\int_1^4 \{\lvert x-1\rvert + \lvert x-2\rvert + \lvert x-4\rvert\}\,dx$$

$$= \int_1^2 \{(x-1)-(x-2)-(x-4)\}\,dx + \int_2^4 \{(x-1)+(x-2)-(x-4)\}\,dx$$

$$= \int_1^2 (5-x)\,dx + \int_2^4 (1+x)\,dx = \left[5x - \frac{x^2}{2}\right]_1^2 + \left[x + \frac{x^2}{2}\right]_2^4$$

$$= (10-2) - \left(5 - \frac{1}{2}\right) + (4+8) - (2+2) = 8 - \frac{9}{2} + 12 - 4 = 16 - \frac{9}{2} = \frac{23}{2}$$

18. Solve the differential equation $(\tan^{-1} x - y)dx = (1 + x^2)dy$.

Sol. Consider $(\tan^{-1} x - y)dx = (1 + x^2)dy$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{\tan^{-1}x - y}{1+x^2} = -\frac{y}{1+x^2} + \frac{\tan^{-1}x}{1+x^2}$$

$$\Rightarrow \qquad \frac{dy}{dx} + \frac{1}{1+x^2}\cdot y = \frac{\tan^{-1}x}{1+x^2}$$

Here $P(x) = \dfrac{1}{1+x^2}$, $Q(x) = \dfrac{\tan^{-1}x}{1+x^2}$

Integrating factor $= e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

Solution is $(I.F.)y = \int (I.F.).Q(x)dx$

$$\Rightarrow \quad e^{\tan^{-1}x}\cdot y = \int e^{\tan^{-1}x}\cdot\frac{\tan^{-1}x}{1+x^2}dx$$

Let $\tan^{-1}x = t$

$\Rightarrow \frac{1}{1+x^2}dx = dt$

$$= \int t.e^t\,dt = t.e^t - \int 1.e^t\,dt$$

$$\Rightarrow \quad e^{\tan^{-1}x}\cdot y = te^t - e^t + C$$

$$\Rightarrow \quad e^{\tan^{-1}x}\cdot y = \tan^{-1}x.e^{\tan^{-1}x} - e^{\tan^{-1}x} + C,$$ C is constant of integration.

$$\Rightarrow \quad y = \tan^{-1}x - 1 + C.e^{-\tan^{-1}x}$$ is the required solution.

19. Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence find the area of the triangle.

Sol. Given points are $A(2\hat{i} - \hat{j} + \hat{k})$, $B(\hat{i} - 3\hat{j} - 5\hat{k})$ and $C(3\hat{i} - 4\hat{j} - 4\hat{k})$

$$\overrightarrow{AB} = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overrightarrow{BC} = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}$$

$$\overrightarrow{CA} = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k}) = -\hat{i} + 3\hat{j} + 5\hat{k}$$

We notice $\overrightarrow{BC}\cdot\overrightarrow{CA} = -2 - 3 + 5 = 0 \quad \Rightarrow \overrightarrow{BC} \perp \overrightarrow{CA}$

$\Rightarrow$ triangle ABC is right-angled at C.

$$\therefore \quad \text{Area of } \Delta = \frac{1}{2}\left|\overrightarrow{CA}\times\overrightarrow{CB}\right| = \frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 3 & 5\\ 2 & -1 & 1\end{matrix}\right\| = \frac{1}{2}\left|8\hat{i} + 11\hat{j} - 5\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{64 + 121 + 25} = \frac{1}{2}\sqrt{210} \text{ sq units}$$

20. Find the value of λ, if four points with position vectors $3\hat{i} + 6\hat{j} + 9\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$, $2\hat{i} + 3\hat{j} + \hat{k}$ and $4\hat{i} + 6\hat{j} + \lambda\hat{k}$ are coplanar.

Sol. Let points be $A(3\hat{i} + 6\hat{j} + 9\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$, $C(2\hat{i} + 3\hat{j} + \hat{k})$ and $D(4\hat{i} + 6\hat{j} + \lambda\hat{k})$

If points are coplanar then $\left[\overrightarrow{AB} \quad \overrightarrow{AC} \quad \overrightarrow{AD}\right] = 0$

$\overrightarrow{AB} = (\hat{i} + 2\hat{j} + 3\hat{k}) - (3\hat{i} + 6\hat{j} + 9\hat{k}) = -2\hat{i} - 4\hat{j} - 6\hat{k}$,

$$\overrightarrow{AC} = (2\hat{i} + 3\hat{j} + \hat{k}) - (3\hat{i} + 6\hat{j} + 9\hat{k}) = -\hat{i} - 3\hat{j} - 8\hat{k},$$

$$\overrightarrow{AD} = (4\hat{i} + 6\hat{j} + \lambda\hat{k}) - (3\hat{i} + 6\hat{j} + 9\hat{k}) = \hat{i} + (\lambda - 9)\hat{k}$$

$$\therefore \quad [\overrightarrow{AB} \quad \overrightarrow{AC} \quad \overrightarrow{AD}] = 0 \Rightarrow \begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & \lambda - 9 \end{vmatrix} = 0$$

$$\Rightarrow -2(-3\lambda + 27) + 4(-\lambda + 9 + 8) - 6(3) = 0$$

$$\Rightarrow \quad 6\lambda - 54 - 4\lambda + 68 - 18 = 0 \Rightarrow 2\lambda - 4 = 0 \Rightarrow \lambda = 2$$

21. There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement. Let *X* denote the sum of the numbers on the two drawn cards. Find the mean and variance of *X*.

Sol. 4 cards are numbered 1, 3, 5 and 7

Total possibilities are ${}^4C_2 = 6$

Total outcomes of combination of 2 cards

(1, 3) total = 4, (3, 5) total = 8

(1, 5) total = 6, (3, 7) total = 10

(1, 7) total = 8, (5, 7) total = 12

X : sum of numbers on two cards

X can take values 4, 6, 8, 10, 12

Table for probability distribution and calculation of mean and variance is as follow:

X	$P(X)$	$X \cdot P(X)$	$X^2P(X)$
4	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{16}{6}$
6	$\frac{1}{6}$	$\frac{6}{6}$	$\frac{36}{6}$
8	$\frac{2}{6}$	$\frac{16}{6}$	$\frac{128}{6}$
10	$\frac{1}{6}$	$\frac{10}{6}$	$\frac{100}{6}$
12	$\frac{1}{6}$	$\frac{12}{6}$	$\frac{144}{6}$
	$\frac{6}{6} = 1$	$\Sigma XP(X) = 8$	$\Sigma X^2P(X) = \frac{424}{6} = \frac{212}{3}$

$$\text{Mean} = \Sigma XP(X) = 8$$

$$\text{Variance} = \Sigma X^2P(X) - \{\Sigma XP(X)\}^2$$

$$= \frac{212}{3} - (8)^2 = \frac{212}{3} - 64 = \frac{212 - 192}{3} = \frac{20}{3}$$

22. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain *A* grade and 10% irregular students attain *A* grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an *A* grade. What is the probability that the student has 100% attendance? Is regularity required only in school? Justify your answer.

Sol. A : students with 100% attendance $\quad$ B : students are irregular

$P(A) = \frac{30}{100} = \frac{3}{10}$ $\quad$ $P(B) = \frac{70}{100} = \frac{7}{10}$

E : a student attained grade A

$P(E/A) = \frac{70}{100} = \frac{7}{10}$ $\quad$ $P(E/B) = \frac{10}{100} = \frac{1}{10}$

Using Bayes' Theorem, student obtaining A grade has 100% attendance

$$P(A/E) = \frac{P(A)P(E/A)}{P(A)P(E/A) + P(B)P(E/B)}$$

$$= \frac{\frac{3}{10} \times \frac{7}{10}}{\frac{3}{10} \times \frac{7}{10} + \frac{7}{10} \times \frac{1}{10}} = \frac{21}{21 + 7} = \frac{21}{28} = \frac{3}{4}$$

Not only in school but all walks of life, unless we are regular some gaps will escape in, which become difficult to fill in with the passage of time, which hamper the growth.

23. Maximise $Z = x + 2y$

subject to the constraints

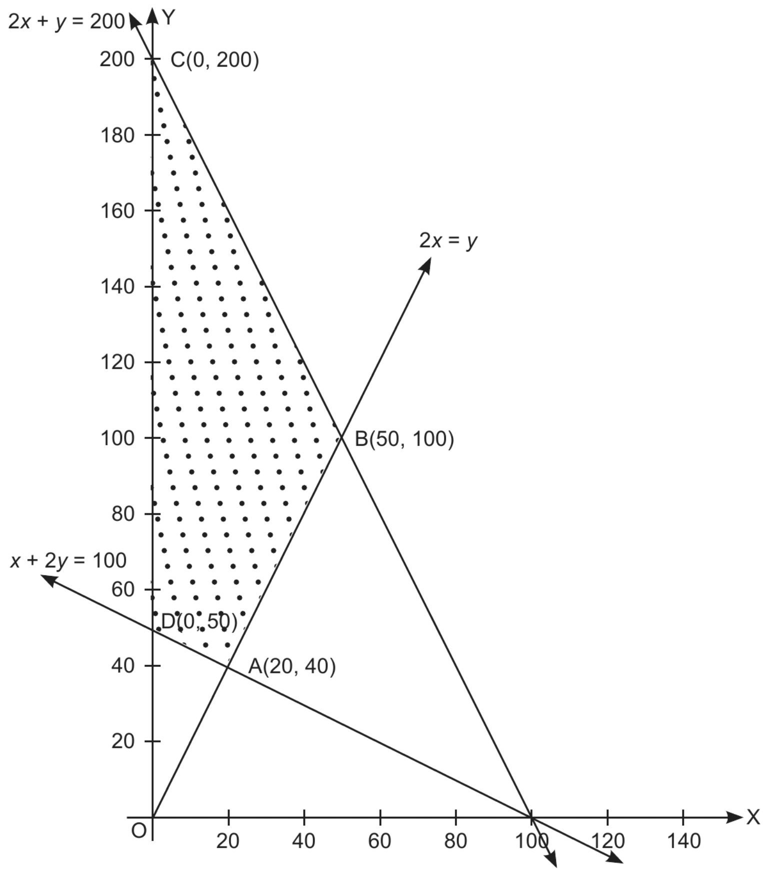
$$x + 2y \geq 100$$
$$2x - y \leq 0$$
$$2x + y \leq 200$$
$$x, y \geq 0$$

Solve the above *LPP* graphically.

Sol. To maximise $Z = x + 2y$

subject to the constraints

$$\begin{aligned} x + 2y &\geq 100 \\ 2x - y &\leq 0 \\ 2x + y &\leq 200 \\ x, y &\geq 0 \end{aligned}$$



On plotting the graph of inequations, we notice shaded portion is feasible solution. Possible points for maximum Z are $A(20, 40)$, $B(50, 100)$, $C(0, 200)$ and $D(0, 50)$.

Points	$Z = x + 2y$	Value	
$A(20, 40)$	20 + 80	100	
$B(50, 100)$	50 + 200	250	
$C(0, 200)$	0 + 400	400	← Maximum
$D(0, 50)$	0 + 100	100	

$\therefore$ Z is maximum for $C(0, 200)$, i.e. $x = 0, y = 200$.

24. Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equations $x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$.

Sol. Consider $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I \qquad ...(i)$$

Consider equations
$$x - y + z = 4$$
$$x - 2y - 2z = 9$$
$$2x + y + 3z = 1$$

Corresponding matrix equation is

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $BX = C$ is matrix equation.

Its solution is $X = B^{-1}C$ ...(*ii*)

From (*i*), we have

$$AB = 8I \Rightarrow \left(\frac{1}{8}A\right)B = I \Rightarrow B^{-1} = \frac{1}{8}A$$

$\therefore$ From (*ii*), $X = \frac{1}{8}AC = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$\therefore$ $x = 3, y = -2, z = -1$ is required solution.

25. Consider $f : R - \left\{-\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ given by $f(x) = \frac{4x+3}{3x+4}$. Show that f is bijective. Find the inverse of f and hence find $f^{-1}(0)$ and x such that $f^{-1}(x) = 2$.

Or

Let $A = Q \times Q$ and let * be a binary operation on A defined by $(a, b) * (c, d) = (ac, b + ad)$ for $(a, b), (c, d) \in A$. Determine, whether * is commutative and associative. Then, with respect to * on A

(*i*) Find the identity element in A.

(*ii*) Find the invertible elements of A.

Sol. Given $f(x) = \frac{4x+3}{3x+4}, f : R - \left\{-\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$

For one-one: Let $x_1, x_2 \in R - \left\{-\frac{4}{3}\right\}$

$$f(x_1) = f(x_2) \Rightarrow \frac{4x_1+3}{3x_1+4} = \frac{4x_2+3}{3x_2+4}$$

$$\Rightarrow 12x_1x_2 + 9x_2 + 16x_1 + 12 = 12x_1x_2 + 9x_1 + 16x_2 + 12$$

$$\Rightarrow 7x_1 = 7x_2 \Rightarrow x_1 = x_2$$

$\therefore$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Hence, one-one.

For onto: Let for $y \in R - \left\{\frac{4}{3}\right\}$, there exists $x \in R - \left\{-\frac{4}{3}\right\}$

Such that $f(x) = y$

$$\Rightarrow \frac{4x+3}{3x+4} = y \Rightarrow 4x + 3 = 3xy + 4y$$

$$\Rightarrow 4x - 3xy = 4y - 3 \Rightarrow x(4 - 3y) = 4y - 3$$

$$\Rightarrow x = \frac{4y-3}{4-3y} \in R - \left\{-\frac{4}{3}\right\}.$$ Hence, onto.

Since, f is one-one and onto. Hence, bijective function.

For inverse:

$$f^{-1}(y) = \frac{4y-3}{4-3y}$$

i.e. $f^{-1}(x) = \frac{4x-3}{4-3x}$ is inverse of 'f'.

$$f^{-1}(0) = \frac{0-3}{4-0} = -\frac{3}{4}$$

$$f^{-1}(x) = 2 \Rightarrow \frac{4x-3}{4-3x} = 2 \Rightarrow 4x - 3 = 8 - 6x$$

$$\Rightarrow 10x = 11 \Rightarrow x = \frac{11}{10}$$

Or

For commutative: Let $(a, b), (c, d) \in A$

$$(a, b) * (c, d) = (ac, b + ad)$$
$$(c, d) * (a, b) = (ac, d + bc)$$

We notice $(ac, b + ad)$ may not be equal to $(ac, d + bc)$

Hence, not commutative.

For example let $(1, 2), (3, 4) \in A$

$$\therefore \quad (1, 2) * (3, 4) = (1 \times 3, 2 + 1 \times 4) = (3, 6)$$
$$(3, 4) * (1, 2) = (3 \times 1, 4 + 3 \times 2) = (3, 10). \text{ Not equal}$$

For associative: Consider $(a, b), (c, d), (e, f) \in A$

$$[(a, b) * (c, d)] * (e, f) = (ac, b + ad) * (e, f)$$
$$= (ace, b + ad + acf)$$

and $$(a, b) * [(c, d) * (e, f)] = (a, b) * (ce, d + cf)$$
$$= (ace, b + ad + acf)$$

As $$[(a, b) * (c, d)] * (e, f) = (a, b) * [(c, d) * (e, f)]$$

Hence, associative.

Given $A = Q \times Q$ and binary operation * as $(a, b)*(c, d) = (ac, b + ad)$

(*i*) **For identity element:** Let $(e_1, e_2) \in A$ is identity element,

then $(a, b) * (e_1, e_2) = (a, b) = (e_1, e_2) * (a, b)$

$$\Rightarrow \quad (ae_1, b + ae_2) = (a, b) = (e_1a, e_2 + e_1b)$$
$$\Rightarrow \quad ae_1 = a \text{ and } b + ae_2 = b$$
$$\Rightarrow \quad e_1 = 1 \text{ and } e_2 = 0$$

$\therefore$ identity element is $(1, 0)$

(*ii*) **For invertible element:** Let $(a, b) \in A$ is invertible and $(x, y) \in A$ is its inverse.

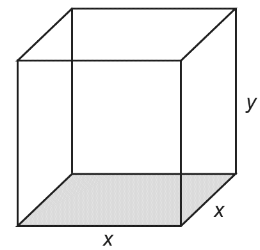
$$\therefore \quad (a, b) * (x, y) = (1, 0) = (x, y) * (a, b)$$
$$\Rightarrow \quad (ax, b + ay) = (1, 0) = (xa, y + xb)$$
$$\Rightarrow \quad ax = 1, b + ay = 0$$
$$\Rightarrow \quad x = \frac{1}{a}, y = -\frac{b}{a}, a \neq 0$$

$(a, b) \in A$ is invertible

if $a \neq 0$ and inverse is $\left(\frac{1}{a}, -\frac{b}{a}\right)$

26. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

Sol. Let x be side of the square base and y be the height of the cuboid



Volume $(V) = x.x.y = x^2y \quad ...(i) \Rightarrow y = \dfrac{V}{x^2}$

Surface area $(S) = 2(x\,.\,x + x\,.\,y + x\,.\,y)$

$$= 2x^2 + 4xy = 2x^2 + 4x\frac{V}{x^2} \qquad \text{[from } (i)\text{]}$$

$$S = 2x^2 + \frac{4V}{x}$$

Differentiating both sides w.r.t. x, we get,

$$\frac{dS}{dx} = 4x - \frac{4V}{x^2}$$

For minimum surface area,

$$\frac{dS}{dx} = 0 \Rightarrow 4x - \frac{4V}{x^2} = 0 \Rightarrow x^3 = V$$

$$\Rightarrow \qquad x = \sqrt[3]{V}$$

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3} \Rightarrow \left.\frac{d^2S}{dx^2}\right]_{x=\sqrt[3]{V}} = 4 + \frac{8V}{V} > 0$$

$\therefore$ for $x = \sqrt[3]{V}$, surface area is minimum

$$\Rightarrow \qquad x^3 = V \Rightarrow x^3 = x^2y \qquad \text{[from } (i)\text{]}$$

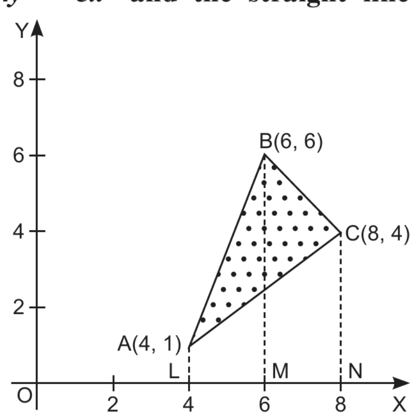
$$\Rightarrow \qquad x = y \Rightarrow \text{cuboid is a cube.}$$

27. Using the method of integration, find the area of the triangle *ABC*, coordinates of whose vertices are *A*(4, 1), *B*(6, 6) and *C*(8, 4).

Or

Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

Sol. Given vertices $A(4, 1)$, $B(6, 6)$, $C(8, 4)$



Equation of AB: $y - 1 = \dfrac{6-1}{6-4}(x-4)$

$$\Rightarrow \qquad y = \frac{5}{2}(x-4) + 1 = \frac{5}{2}x - 9$$

Equation of BC: $y - 4 = \dfrac{4-6}{8-6}(x-8)$

$$\Rightarrow \qquad y = -x + 8 + 4 = -x + 12$$

$$\text{Equation of } AC: \; y - 4 = \frac{4-1}{8-4}(x-8)$$

$$\Rightarrow \quad y = \frac{3}{4}x - 2$$

$$\text{Area } (\Delta ABC) = \int_4^6 y_{AB}\,dx + \int_6^8 y_{BC}\,dx - \int_4^8 y_{AC}\,dx$$

$$= \int_4^6 \left(\frac{5}{2}x - 9\right)dx + \int_6^8 (-x + 12)\,dx - \int_4^8 \left(\frac{3}{4}x - 2\right)dx$$

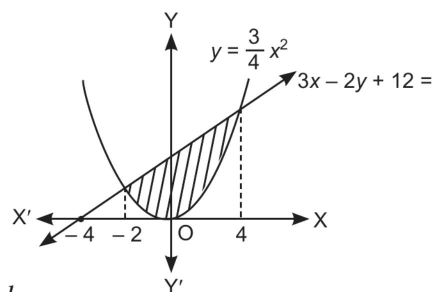
$$= \left[\frac{5x^2}{4} - 9x\right]_4^6 + \left[-\frac{x^2}{2} + 12x\right]_6^8 - \left[\frac{3x^2}{8} - 2x\right]_4^8$$

$$= (45-54)-(20-36)+(-32+96)-(-18+72)-(24-16)+(6-8)$$

$$= -9 + 16 + 64 - 54 - 8 - 2 = 7 \text{ sq units}$$

Or

Eliminating y from the equations, we get



$$3x - \frac{3}{2}x^2 + 12 = 0$$

$$\Rightarrow \quad x^2 - 2x - 8 = 0$$

$$\Rightarrow \quad (x-4)(x+2) = 0$$

$$\Rightarrow \quad x = -2, 4$$

$$\text{Area} = \int_{-2}^{4} \left\{\frac{3x+12}{2} - \frac{3}{4}x^2\right\} dx$$

$$= \left[\frac{3x^2}{4} + 6x - \frac{x^3}{4}\right]_{-2}^{4}$$

$$= [(12 + 24 - 16) - (3 - 12 + 2)] \text{ sq units}$$

$$= (20 + 7) = 27 \text{ sq units}$$

28. Find the particular solution of the differential equation $(x - y)\dfrac{dy}{dx} = x + 2y$, given that $y = 0$ when $x = 1$.

Sol. Consider equation $(x-y)\dfrac{dy}{dx} = x + 2y$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x + 2y}{x - y} \qquad ...(i)$$

Let $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

From (i), we get

$$v + x\frac{dv}{dx} = \frac{x + 2vx}{x - vx} = \frac{1 + 2v}{1 - v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

$$= \frac{1+2v-v+v^2}{1-v} = \frac{1+v+v^2}{1-v}$$

$$\Rightarrow \quad \frac{1-v}{1+v+v^2}dv = \frac{dx}{x}$$

Integrating we get

$$\int \frac{1-v}{1+v+v^2}dv = \int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{2-2v}{1+v+v^2}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{3-(1+2v)}{1+v+v^2}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\left[3\int \frac{1}{1+v+v^2}dv - \int \frac{1+2v}{1+v+v^2}dv\right] = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{3}{2}\int \frac{1}{\left(v+\frac{1}{2}\right)^2+\frac{3}{4}}dv - \frac{1}{2}\int \frac{1+2v}{1+v+v^2}dv = \int \frac{dx}{x}$$

$$\Rightarrow \frac{3}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{v+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) - \frac{1}{2}\log|1+v+v^2| = \log|x| + C$$

$$\Rightarrow \quad \sqrt{3}\tan^{-1}\left(\frac{\frac{2y+x}{x}}{\sqrt{3}}\right) - \frac{1}{2}\log\left|1+\frac{y}{x}+\frac{y^2}{x^2}\right| = \log|x| + C$$

$$\Rightarrow \quad \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) - \frac{1}{2}\log|x^2+xy+y^2| = C \qquad ...(i)$$

Given $y = 0$, when $x = 1$

$$\Rightarrow \quad \sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \frac{1}{2}\log|1| = C$$

$$\Rightarrow \quad \sqrt{3}\cdot\frac{\pi}{6} = C \quad \Rightarrow \quad C = \frac{\sqrt{3}\pi}{6}$$

Substituting in (i), we get

$$\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)-\frac{1}{2}\log|x^2+xy+y^2| = \frac{\sqrt{3}\pi}{6} \text{ is required solution.}$$

29. Find the coordinates of the point where the line through the points (3, –4, –5) and (2, –3, 1), crosses the plane determined by the points (1, 2, 3), (4, 2, –3) and (0, 4, 3).

Or

A variable plane which remains at a constant distance 3*p* from the origin cuts the coordinate axes at *A*, *B*, *C*. Show that the locus of the centroid of triangle *ABC* is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$

Sol. Equations of line through the points (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}, \text{ i.e. } \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda \text{ (say)}$$

General point on line is $(-\lambda+3, \lambda-4, 6\lambda-5)$ ...(i)

Equation of a plane through the points (1, 2, 3), (4, 2, – 3) and (0, 4, 3) is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 4-1 & 2-2 & -3-3 \\ 0-1 & 4-2 & 3-3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-1 & y-2 & z-3 \\ 3 & 0 & -6 \\ -1 & 2 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-1)(12)-(y-2)(-6)+(z-3)(6) = 0$$

$$\Rightarrow \quad 12x-12+6y-12+6z-18 = 0$$

$$\Rightarrow \quad 12x+6y+6z-42 = 0$$

$$\Rightarrow \quad 2x+y+z-7 = 0$$

If point (i), lies in plane then

$$-2\lambda+6+\lambda-4+6\lambda-5-7 = 0$$

$$\Rightarrow \quad 5\lambda-10 = 0 \Rightarrow \lambda = 2$$

Substituting in (i), point of intersection is $(-2+3, 2-4, 12-5)$, i.e. $(1, -2, 7)$

Or

Let plane cut off intercepts a, b and c on x, y and z axes respectively.

Then equation of plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \quad \text{...}(i)$$

and coordinates of A, B, C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

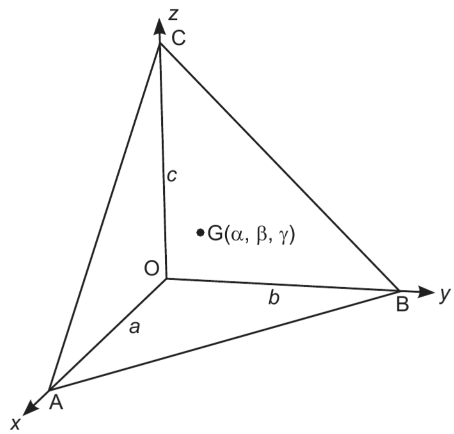
Let centroid of triangle ABC be $G(\alpha, \beta, \gamma)$

$$\therefore \quad \left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right) = (\alpha, \beta, \gamma)$$

$\Rightarrow a = 3\alpha, b = 3\beta, c = 3\gamma$

From (i), we get plane as

$$\frac{x}{3\alpha} + \frac{y}{3\beta} + \frac{z}{3\gamma} = 1$$



Distance of plane from origin is $3p$

$$\therefore \quad \left|\frac{\frac{1}{3\alpha}\cdot 0 + \frac{1}{3\beta}\cdot 0 + \frac{1}{3\gamma}\cdot 0 - 1}{\sqrt{\frac{1}{9\alpha^2} + \frac{1}{9\beta^2} + \frac{1}{9\gamma^2}}}\right| = 3p$$

$$\Rightarrow \quad \frac{1}{\frac{1}{9\alpha^2} + \frac{1}{9\beta^2} + \frac{1}{9\gamma^2}} = 9p^2$$

$$\Rightarrow \quad \frac{1}{9\alpha^2} + \frac{1}{9\beta^2} + \frac{1}{9\gamma^2} = \frac{1}{9p^2}$$

$$\Rightarrow \quad \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{p^2}$$

$\therefore$ Locus of centroid $G(\alpha, \beta, \gamma)$ is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$$

[SET II: UNCOMMON QUESTIONS TO SET I]

12. The length x, of a rectangle is decreasing at the rate of 5 cm/minute and the width y, is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rate of change of the area of the rectangle.

Sol. Area of the rectangle $A = xy$, $\frac{dx}{dt} = -5$ cm/min, $\frac{dy}{dt} = 4$ cm/min

$$\therefore \quad \frac{dA}{dt} = \frac{d}{dt}(xy) = x\frac{dy}{dt} + y\frac{dx}{dt} = 4x - 5y$$

$$\left.\frac{dA}{dt}\right]_{x=8 \text{ and } y=6} = 32 - 30 = 2\text{cm}^2/\text{min}$$

Hence, area is increasing at the rate of 2cm^2/min.

20. Find: $\displaystyle\int \frac{\sin\theta}{(4+\cos^2\theta)(2-\sin^2\theta)}d\theta$

Sol. Consider $\displaystyle\int \frac{\sin\theta}{(4+\cos^2\theta)(2-\sin^2\theta)}d\theta = \int \frac{\sin\theta}{(4+\cos^2\theta)(1+\cos^2\theta)}d\theta$

$$= -\int \frac{1}{(4+t^2)(1+t^2)}dt \qquad \begin{vmatrix}\text{Let } \cos\theta = t \\ \Rightarrow -\sin\theta d\theta = dt\end{vmatrix}$$

$$= -\frac{1}{3}\int \frac{(4+t^2)-(1+t^2)}{(4+t^2)(1+t^2)}dt$$

$$= -\frac{1}{3}\left[\int \frac{1}{1+t^2}dt - \int \frac{1}{4+t^2}dt\right]$$

$$= -\frac{1}{3}\left[\tan^{-1}t - \frac{1}{2}\tan^{-1}\frac{t}{2}\right] + C$$

$$= -\frac{1}{3}\left[\tan^{-1}(\cos\theta) - \frac{1}{2}\tan^{-1}\left(\frac{\cos\theta}{2}\right)\right] + C$$

21. Solve the following linear programming problem graphically:

Maximise $Z = 34x + 45y$

under the following constraints

$$x + y \le 300$$
$$2x + 3y \le 70$$
$$x \ge 0, y \ge 0$$

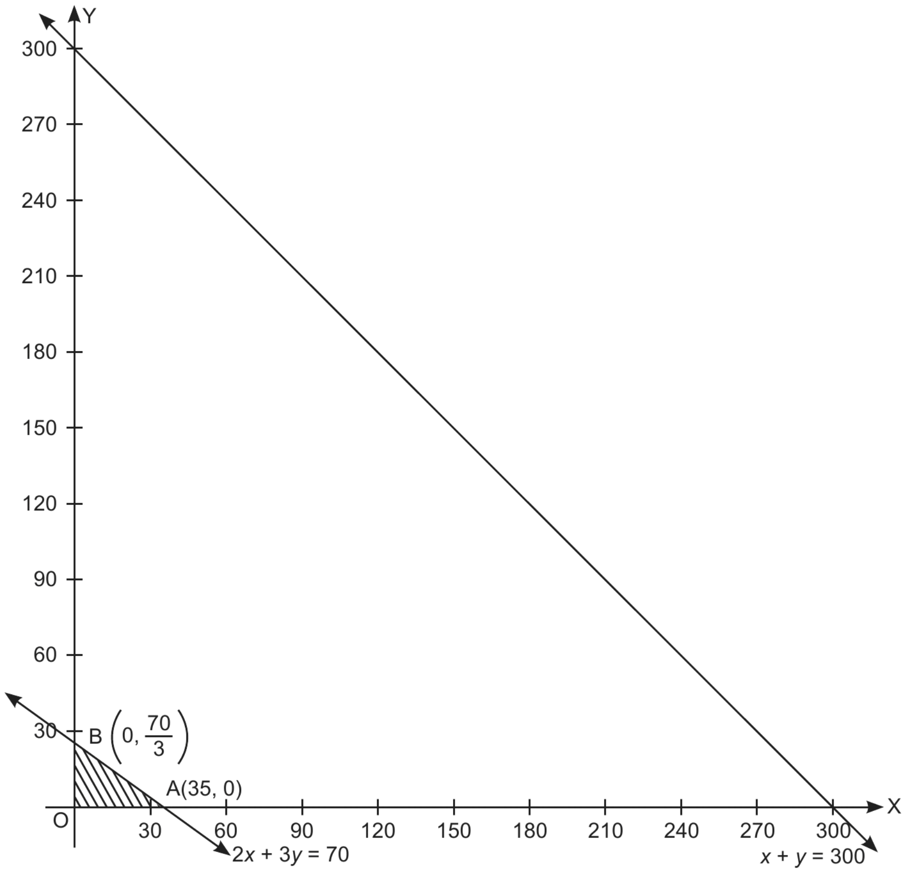
Sol. To maximise $Z = 34x + 45y$

subject to the constraints

$$x + y \le 300$$
$$2x + 3y \le 70$$
$$x \ge 0, y \ge 0$$

Plotting the graph of the inequations we notice shaded portion is feasible solution.

Possible points for maximum Z are $A(35, 0)$, $B\left(0, \frac{70}{3}\right)$



Points	$Z = 34x + 45y$	Value	
$A(35, 0)$	1190 + 0	1190	← Maximum
$B\left(0, \frac{70}{3}\right)$	0 + 1050	1050	

Z is maximum for $A(35, 0)$, i.e. $x = 35$ and $y = 0$

22. Find the value of x such that the points $A(3, 2, 1)$, $B(4, x, 5)$, $C(4, 2, -2)$ and $D(6, 5, -1)$ are coplanar.

Sol. If points $A(3, 2, 1)$, $B(4, x, 5)$, $C(4, 2, -2)$ and $D(6, 5, -1)$ are coplanar then

$\left[\overrightarrow{AB} \quad \overrightarrow{AC} \quad \overrightarrow{AD}\right] = 0$

$$\overrightarrow{AB} = (4\hat{i} + x\hat{j} + 5\hat{k}) - (3\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} + (x - 2)\hat{j} + 4\hat{k},$$

$$\overrightarrow{AC} = (4\hat{i} + 2\hat{j} - 2\hat{k}) - (3\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{k},$$

$$\overrightarrow{AD} = (6\hat{i} + 5\hat{j} - \hat{k}) - (3\hat{i} + 2\hat{j} + \hat{k}) = 3\hat{i} + 3\hat{j} - 2\hat{k}$$

$$\left[\overrightarrow{AB} \quad \overrightarrow{AC} \quad \overrightarrow{AD}\right] = \begin{vmatrix} 1 & x-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$$\Rightarrow \quad 1(9) - (x-2)(7) + 4(3) = 0$$

$$\Rightarrow \quad 9 - 7x + 14 + 12 = 0$$

$$\Rightarrow \quad 7x = 35 \Rightarrow x = 5$$

23. Find the general solution of the differential equation $y\,dx - (x + 2y^2)\,dy = 0$.

Sol. Consider $ydx - (x + 2y^2)dy = 0$

$$\Rightarrow \frac{dx}{dy} = \frac{x + 2y^2}{y} = \frac{x}{y} + 2y$$

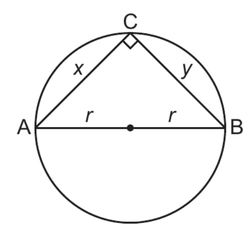
$$\Rightarrow \frac{dx}{dy} - \frac{1}{y}\cdot x = 2y$$

Integrating factor $(I.F.) = e^{-\int \frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$

Solution is, $\frac{1}{y}\cdot x = \int \frac{1}{y}\cdot 2y\,dy \Rightarrow \frac{x}{y} = 2y + C \Rightarrow x = 2y^2 + Cy$

28. *AB* is the diameter of a circle and *C* is any point on the circle. Show that the area of triangle *ABC* is maximum, when it is an isosceles triangle.

Sol. Let radius of given circle be r,



and for triangle ABC, $AC = x,\ BC = y$

$$\therefore \quad x^2 + y^2 = (2r)^2$$

$$\Rightarrow \quad y = \sqrt{4r^2 - x^2} \quad \text{...}(i)$$

$$\text{Area of } \Delta ABC = A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{4r^2 - x^2} \quad \{\text{from } (i)\}$$

$$\frac{dA}{dx} = \frac{1}{2}\left[x\cdot\frac{1}{2\sqrt{4r^2 - x^2}}(-2x) + \sqrt{4r^2 - x^2}\right]$$

$$= \frac{1}{2}\left[\frac{-x^2 + 4r^2 - x^2}{\sqrt{4r^2 - x^2}}\right] = \left[\frac{2r^2 - x^2}{\sqrt{4r^2 - x^2}}\right]$$

For maximum area, $\frac{dA}{dx} = 0$

$$\Rightarrow \quad 2r^2 - x^2 = 0$$

$$\Rightarrow \quad x = \sqrt{2}r$$

$$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2 - x^2}\cdot(-2x) - (2r^2 - x^2)\cdot\frac{1}{2\sqrt{4r^2 - x^2}}(-2x)}{(4r^2 - x^2)}$$

$$\left.\frac{d^2A}{dx^2}\right]_{x = \sqrt{2}r} = \frac{\sqrt{4r^2 - 2r^2}\cdot(-2\sqrt{2}r) - 0}{(4r^2 - 2r^2)} < 0$$

$\therefore$ Area is maximum for $x = \sqrt{2}r$

Substituting in (i), we get $y = \sqrt{4r^2 - (\sqrt{2}r)^2}$

$$= \sqrt{4r^2 - 2r^2} = \sqrt{2r^2} = \sqrt{2}r$$

$\Rightarrow$ $x = y$. Hence, area is maximum when triangle is an isosceles.

29. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Hence using A^{-1} solve the system of equations $2x - 3y + 5z = 11, 3x + 2y - 4z = -5, x + y - 2z = -3.$

Sol. $$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2(0) + 3(-2) + 5(1) = -1 \neq 0$$

Hence, A^{-1} exists.

Cofactors of elements of $|A|$ are

$A_{11} = 0,\quad A_{12} = 2,\quad A_{13} = 1$

$A_{21} = -1,\quad A_{22} = -9,\quad A_{23} = -5$

$A_{31} = 2,\quad A_{32} = 23,\quad A_{33} = 13$

$$\text{Adj}\, A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\, A = -\frac{1}{1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \qquad ...(i)$$

Consider equations,

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Corresponding matrix equation is

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

i.e., $AX = B$, Its solution is $X = A^{-1}B$

$$X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \qquad \text{[from } (i)\text{]}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad x = 1, \ y = 2, \ z = 3$ is solution.

[SET III: UNCOMMON QUESTIONS TO SET I & II]

12. The volume of a sphere is increasing at the rate of 8 cm³/s. Find the rate at which its surface area is increasing when the radius of the sphere is 12 cm.

Sol. Let V, S and r be volume, surface area and radius of a sphere at any time t.

Given $\frac{dV}{dt} = 8 \text{ cm}^3/\text{s}$

To find $\left.\frac{dS}{dt}\right]_{r=12} = ?$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad 8 = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{2}{\pi r^2} \qquad ...(i)$$

Consider $S = 4\pi r^2$

$$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{2}{\pi r^2} \qquad \{\text{from } (i)\}$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{16}{r}$$

$$\therefore \quad \left.\frac{dS}{dt}\right]_{r=12} = \frac{16}{12} = \frac{4}{3} \text{ cm}^2/\text{s}$$

Surface area is increasing at rate of $\frac{4}{3}$ cm²/s

20. Solve the following linear programming problem graphically:

Maximise $Z = 7x + 10y$

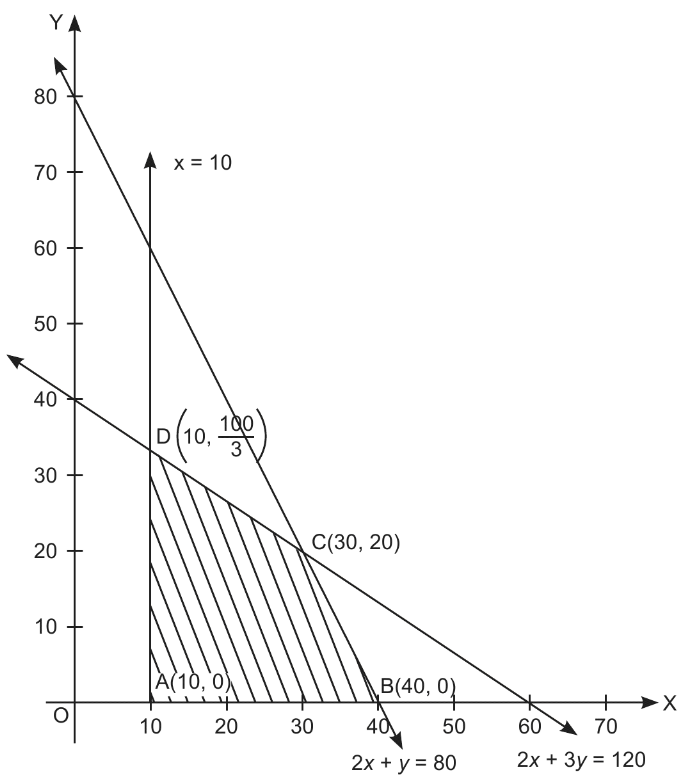
subject to the constraints

$$4x + 6y \le 240$$
$$6x + 3y \le 240$$
$$x \ge 10, \; x \ge 0, \; y \ge 0$$

Sol. To maximise $Z = 7x + 10y$

subject to the constraints

$$4x + 6y \le 240$$
$$\Rightarrow \quad 2x + 3y \le 120$$
$$6x + 3y \le 240$$
$$\Rightarrow \quad 2x + y \le 80$$
$$x \ge 10, \; x \ge 0, \; y \ge 0$$



On plotting the graph of the inequations we notice shaded portion is feasible solution.

Possible points for maximum Z are $A(10, 0)$, $B(40, 0)$, $C(30, 20)$ and $D\left(10, \frac{100}{3}\right)$

Points	$Z = 7x + 10y$	Value	
$A(10, 0)$	$70 + 0$	70	
$B(40, 0)$	$280 + 0$	280	
$C(30, 20)$	$210 + 200$	410	← Maximum
$D\left(10, \frac{100}{3}\right)$	$70 + \frac{1000}{3}$	403.3	

Z is maximum at $C(30, 20)$, i.e. $x = 30$ and $y = 20$

21. Find: $\int \frac{e^x}{(e^x - 1)^2 (e^x + 2)} dx$

Sol. Consider $\int \frac{e^x}{(e^x-1)^2(e^x + 2)} dx = \int \frac{1}{(t-1)^2(t+2)} dt$ $\quad\Big|$ Let $e^x = t \Rightarrow e^x\, dx = dt$

Let $$\frac{1}{(t-1)^2(t+2)} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+2} \qquad ...(i)$$

$$\Rightarrow \quad 1 = A(t-1)(t+2) + B(t+2) + C(t-1)^2$$
$$= A(t^2 + t - 2) + B(t + 2) + C(t^2 - 2t + 1)$$
$$= t^2(A + C) + t(A + B - 2C) + (-2A + 2B + C)$$

Comparing the coefficients, we get

$$A + C = 0 \Rightarrow A = -C$$
$$A + B - 2C = 0 \Rightarrow -3A = B$$
$$-2A + 2B + C = 1 \Rightarrow -2A - 6A - A = 1 \Rightarrow A = -\frac{1}{9}$$
$$\Rightarrow \quad A = -\frac{1}{9}, B = \frac{1}{3}, C = \frac{1}{9}$$

Substituting in (i) and integrating, we get

$$\int \frac{1}{(t-1)^2(t+2)} dt = -\frac{1}{9}\int \frac{1}{t-1} dt + \frac{1}{3}\int \frac{1}{(t-1)^2} dt + \frac{1}{9}\int \frac{1}{t+2} dt$$
$$= -\frac{1}{9} \log|t-1| - \frac{1}{3(t-1)} + \frac{1}{9}\log|t+2| + C$$
$$\int \frac{e^x}{(e^x-1)^2(e^x+2)} dx = -\frac{1}{9} \log|e^x - 1| - \frac{1}{3(e^x-1)} + \frac{1}{9}\log|e^x + 2| + C$$

22. If $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 2\hat{j} - 3\hat{k}$, then express $\vec{b}$ in the form of $\vec{b} = \vec{b_1} + \vec{b_2}$, where $\vec{b_1}$ is parallel to $\vec{a}$ and $\vec{b_2}$ is perpendicular to $\vec{a}$.

Sol. Given $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 7i + 2j - 3k$

Also $\vec{b} = \vec{b_1} + \vec{b_2}$ where $\vec{b_1} \parallel \vec{a}$ and $\vec{b_2} \perp \vec{a}$ ...(*i*)

As $\vec{b_1} \parallel \vec{a} \Rightarrow \vec{b_1} = \lambda\vec{a} \Rightarrow \vec{b_1} = \lambda(2\hat{i} - \hat{j} - 2\hat{k})$ ...(*ii*)

From (*i*) $\vec{b} = \vec{b_1} + \vec{b_2} \Rightarrow 7\hat{i} + 2\hat{j} - 3\hat{k} = \lambda(2\hat{i} - \hat{j} - 2\hat{k}) + \vec{b_2}$

$\Rightarrow$ $\vec{b_2} = (7 - 2\lambda)\hat{i} + (2 + \lambda)\hat{j} + (-3 + 2\lambda)\hat{k}$ ...(*iii*)

Also $\vec{b_2} \perp \vec{a} \Rightarrow \vec{b_2} \cdot \vec{a} = 0 \Rightarrow 2(7 - 2\lambda) - 1(2 + \lambda) - 2(-3 + 2\lambda) = 0$

$\Rightarrow 14 - 4\lambda - 2 - \lambda + 6 - 4\lambda = 0 \Rightarrow 9\lambda = 18 \Rightarrow \lambda = 2$

$\therefore$ From (*ii*) and (*iii*), on substituting for λ, we get

$$\vec{b_1} = 4\hat{i} - 2\hat{j} - 4\hat{k} \text{ and } \vec{b_2} = 3\hat{i} + 4\hat{j} + \hat{k}$$

23. Find the general solution of the differential equation $\frac{dy}{dx} - y = \sin x$.

Sol. Consider equation $\frac{dy}{dx} - y = \sin x$

Here $P(x) = -1$, $Q(x) = \sin x$

Integrating factor $(I.F.) = e^{-\int dx} = e^{-x}$

$\therefore$ Solution is $(I.F.)y = \int\{(I.F.)Q(x)\}dx$

$$e^{-x}.y = \int e^{-x}\sin x\, dx \quad ...(i)$$

Consider, $I = \int \underset{①}{e^{-x}} \underset{②}{\sin x}\, dx$

$$= e^{-x}.(-\cos x) - \int\{-e^{-x}\cdot(-\cos x)\}dx$$

$\Rightarrow$ $I = -e^{-x}\cos x - \int \underset{①}{e^{-x}} \underset{②}{\cos x}\, dx$

$$= -e^{-x}\cos x - \left[e^{-x}\sin x - \int(-e^{-x}).\sin x\, dx\right]$$

$$I = -e^{-x}\cos x - e^{-x}\sin x - I$$

$$\Rightarrow \qquad 2I = -e^x(\cos x + \sin x)$$

$$\Rightarrow \qquad I = \frac{-e^x}{2}(\cos x + \sin x)$$

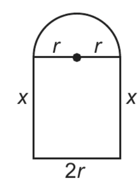
Substituting in (i), we get

$$e^{-x}.y = -\frac{e^{-x}}{2}(\cos x + \sin x) + C$$

$$\Rightarrow \qquad y = -\frac{1}{2}(\cos x + \sin x) + Ce^x \text{ is required solution.}$$

29. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Sol. For the greatest possible light, area of window should be maximum.



Let x be side of a rectangle and r be the radius of semicircle.

$$\therefore \qquad 2x + 2r + \pi r = 10 \Rightarrow 2x + (2 + \pi) r = 10 \qquad ...(i)$$

Area of the window, $A = 2xr + \frac{1}{2}\pi r^2$

$$\Rightarrow \qquad A = r[10 - (2 + \pi) r] + \frac{1}{2}\pi r^2 = 10r - (2 + \pi) r^2 + \frac{1}{2}\pi r^2$$

$$\frac{dA}{dr} = 10 - 2(2 + \pi) r + \pi r = 10 - (4 + 2\pi - \pi) r$$

$$\Rightarrow \qquad \frac{dA}{dr} = 10 - (4 + \pi) r$$

For maximum area $\frac{dA}{dr} = 0 \Rightarrow 10 - (4 + \pi) r = 0 \Rightarrow r = \frac{10}{4+\pi}$

$$\frac{d^2A}{dr^2} = -(4 + \pi), \left.\frac{d^2A}{dr^2}\right]_{r = \frac{10}{4+\pi}} = -(4 + \pi) < 0$$

$\therefore$ for $r = \frac{10}{4+\pi}$ m, area is maximum. Substituting the value of r in (i), we get $x = \frac{10}{4+\pi}$ m.

$\therefore$ dimensions of the rectangle are $\frac{20}{4+\pi}$m, $\frac{10}{4+\pi}$m.

Examination Papers, 2017
[Foreign Set-I, II, III]

1. If $|A| = 3$ and $A^{-1} = \begin{bmatrix} 3 & -1 \\ -\frac{5}{3} & \frac{2}{3} \end{bmatrix}$, then write the adj A.

Sol. $$A^{-1} = \frac{1}{|A|}\text{adj } A \Rightarrow |A|\,A^{-1} = \text{adj } A$$

$$\Rightarrow \quad |A|\begin{bmatrix} 3 & -1 \\ -5/3 & 2/3 \end{bmatrix} = \text{Adj } A$$

$$\Rightarrow \quad \text{Adj } A = 3\begin{bmatrix} 3 & -1 \\ -5/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ -5 & 2 \end{bmatrix}$$

2. For what value of 'k' is the function $f(x) = \begin{cases} \dfrac{\sin 5x}{3x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ continuous at $x = 0$?

Sol. If continuous at $x = 0$, $\lim_{x \to 0} f(x) = f(0)$

$$\Rightarrow \quad \lim_{x \to 0}\left[\frac{\sin 5x}{3x} + \cos x\right] = k$$

$$\Rightarrow \quad \frac{5}{3}\cdot\lim_{x \to 0}\frac{\sin 5x}{5x} + \lim_{x \to 0}\cos x = k \Rightarrow \frac{5}{3} + 1 = k$$

$$\Rightarrow \quad k = \frac{8}{3}$$

3. Evaluate: $\int_0^{2\pi} \cos^5 x\, dx$

Sol. $$\int_0^{2\pi} \cos^5 x\, dx = \int_0^{2\pi} (1 - \sin^2 x)^2 \cos x\, dx$$

| Let $\sin x = t$
| $\Rightarrow \cos x\, dx = dt$
| When $x = 0, t = 0$
| and when $x = 2\pi, t = 0$

$$\therefore \quad \int_0^0 (1 - t^2)^2 dt = 0$$

4. Write the distance of the point (3, – 5, 12) from x-axis.

Sol. Distance of the point (3, – 5, 12) from x-axis $= \sqrt{(-5)^2 + (12)^2} = \sqrt{25 + 144}$

$$= \sqrt{169} = 13 \text{ units}$$

5. If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then find the value of $|2AB|$.

Sol. We have $|2AB| = 2^3|A||B| = 8(-1)(3) = -24$

6. The radius r of a right circular cylinder is decreasing at the rate of 3 cm/min. and its height h is increasing at the rate of 2 cm/min. When r = 7 cm and h = 2 cm, find the rate of change of the volume of cylinder. [Use $\pi = \frac{22}{7}$]

Sol. Let r be radius of base and h be height at time t

$\therefore \quad \frac{dr}{dt} = -3$ cm/min (decreasing) and $\frac{dh}{dt} = 2$ cm/min (increasing)

Volume of cylinder, $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right] = \pi[2r^2 - 6rh]$$

$$\therefore \quad \left.\frac{dV}{dt}\right]_{\substack{r = 7\text{ cm and}\\ h = 2\text{ cm}}} = \frac{22}{7}(2 \times 49 - 6 \times 7 \times 2) = 22(14 - 12) = 44\text{ cm}^3/\text{min}$$

Hence, the rate of change of the volume of cylinder is 44 cm^3/min (increasing).

7. Find $\frac{dy}{dx}$ at $t = \frac{2\pi}{3}$ when $x = 10(t - \sin t)$ and $y = 12(1 - \cos t)$.

Sol. Given

$$x = 10(t - \sin t) \Rightarrow \frac{dx}{dt} = 10(1 - \cos t)$$

$$y = 12(1 - \cos t) \Rightarrow \frac{dy}{dt} = 12\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{12\sin t}{10(1-\cos t)} = \frac{12 \times 2\sin\frac{t}{2}\cos\frac{t}{2}}{10 \times 2\sin^2\frac{t}{2}} = \frac{6}{5}\cot\frac{t}{2}$$

$$\left.\frac{dy}{dx}\right]_{t = \frac{2\pi}{3}} = \frac{6}{5}\cot\frac{\pi}{3} = \frac{6}{5} \times \frac{1}{\sqrt{3}} = \frac{6}{5\sqrt{3}}$$

8. Show that the function f given by $f(x) = \tan^{-1}(\sin x + \cos x)$ is decreasing for all $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Sol. Consider

$$f(x) = \tan^{-1}(\sin x + \cos x)$$

$$\Rightarrow \quad f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot (\cos x - \sin x) \qquad \text{...}(i)$$

Sign of $f'(x)$ depends upon $(\cos x - \sin x)$

We know for $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, $\cos x < \sin x \quad \Rightarrow \quad \cos x - \sin x < 0$

$\Rightarrow \quad f'(x) < 0$ [from (i)]

$\therefore \quad f$ is decreasing for $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

9. A line passes through the point with position vector $2\hat{i} - 3\hat{j} + 4\hat{k}$ and is perpendicular to the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 5\hat{k}) = 7$. Find the equation of the line in cartesian and vector forms.

Sol. Line passes through the point with position vector $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and is perpendicular to the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 5\hat{k}) = 7$, therefore, line is along the vector $\vec{n} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

$\therefore$ Equation of plane in vector form $\vec{r} = (2\hat{i} - 3j + 4k) + \lambda(3i + 4j - 5k)$ and in Cartesian form $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-4}{-5}$.

10. If $P(A) = 0.4$, $P(B) = p$, $P(A \cup B) = 0.6$ and A and B are given to be independent events, find the value of 'p'.

Sol.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)P(B) \quad [\because \text{Events } A \text{ and } B \text{ are independent}]$$

$$\Rightarrow \quad 0.6 = 0.4 + p - 0.4p$$

$$\Rightarrow \quad 0.2 = 0.6p \Rightarrow p = \frac{1}{3}$$

11. A company produces two types of goods A and B, that require gold and silver. Each unit of type A requires 3 g of silver and 1 g of gold while that of type B requires 1 g of silver and 2 g of gold. The company can procure a maximum of 9 g of silver and 8 g of gold. If each unit of type A brings a profit of ₹ 40 and that of type B ₹ 50, formulate *LPP* to maximize profit.

Sol. Let x units of goods A and y units of goods B be produced.

Then *LPP* is to maximise profit $Z = 40x + 50y$

subject to the constraints

$$3x + y \leq 9$$

$$x + 2y \leq 8$$

$$x \geq 0, \ y \geq 0$$

12. Find: $\int \frac{dx}{\sqrt{3 - 2x - x^2}}$

Sol. Consider $\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{4-(1+2x+x^2)}}$

$$= \int \frac{dx}{\sqrt{4-(x+1)^2}} = \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

13. Find the value of $\cot \frac{1}{2}\left[\cos^{-1}\frac{2x}{1+x^2} + \sin^{-1}\frac{1-y^2}{1+y^2}\right]$, $|x| < 1, y > 0$ and $xy < 1$.

Sol. Consider $\cot \frac{1}{2}\left[\cos^{-1}\frac{2x}{1+x^2} + \sin^{-1}\frac{1-y^2}{1+y^2}\right]$

$$= \cot \frac{1}{2}\left[\frac{\pi}{2} - \sin^{-1}\frac{2x}{1+x^2} + \frac{\pi}{2} - \cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$= \cot \frac{1}{2}[\pi - 2\tan^{-1}x - 2\tan^{-1}y] = \cot\left[\frac{\pi}{2} - (\tan^{-1}x + \tan^{-1}y)\right]$$

$$= \tan(\tan^{-1}x + \tan^{-1}y) = \tan\left[\tan^{-1}\frac{x+y}{1-xy}\right] = \frac{x+y}{1-xy}$$

14. Using properties of determinants show that

$$\begin{vmatrix} 1 & 1 & 1+x \\ 1 & 1+y & 1 \\ 1+z & 1 & 1 \end{vmatrix} = xyz + yz + zx + xy.$$

Or

Find matrix X so that $X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$.

Sol. Consider $\begin{vmatrix} 1 & 1 & 1+x \\ 1 & 1+y & 1 \\ 1+z & 1 & 1 \end{vmatrix}$

$$= \begin{vmatrix} 0 & -y & x \\ -z & y & 0 \\ 1+z & 1 & 1 \end{vmatrix} \quad \text{[On performing } R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3\text{]}$$

$$= \begin{vmatrix} 0-x-xz & -y-x & 0 \\ -z & y & 0 \\ 1+z & 1 & 1 \end{vmatrix} \quad \text{[On performing } R_1 \to R_1 - xR_3\text{]}$$

$= [y(-x-xz) - (-z)(-y-x)]$ [On expanding along C_3]

$= -xy - xyz - yz - xz = -(xy + yz + zx + xyz)$

Note: Negative sign is missing in the question.

Or

$$X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}.$$

For multiplication to define, X should be of order 2×2.

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

Using equality of matrices, we get

$a + 4b = -7, 2a + 5b = -8$ and $3a + 6b = -9$

Solving first two equations, we get $a = 1, b = -2$. This also satisfies the third equation.

Also $c + 4d = 2, 2c + 5d = 4$ and $3c + 6d = 6$

Solving first two equations, we get $c = 2, d = 0$

Which satisfies third equation also.

Hence $a = 1, b = -2, c = 2$ and $d = 0$. Hence, $X = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$

15. If $xy = e^{(x-y)}$, then show that $\frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$.

Or

If $\log y = \tan^{-1} x$, then show that $(1 + x^2)\frac{d^2y}{dx^2} + (2x - 1)\frac{dy}{dx} = 0$.

Sol. Consider, $xy = e^{x-y}$

Taking log of both sides, we get

$$\log(xy) = (x-y)\log e \quad \Rightarrow \quad \log(xy) = x - y$$

$$\Rightarrow \quad \log x + \log y = x - y$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{x} + \frac{1}{y}\cdot\frac{dy}{dx} = 1 - \frac{dy}{dx} \quad \Rightarrow \quad \left(\frac{1}{y} + 1\right)\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \quad \left(\frac{1+y}{y}\right)\frac{dy}{dx} = \frac{x-1}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y(x-1)}{x(1+y)}$$

Or

Consider, $\log y = \tan^{-1}x$

Differentiating both sides w.r.t. x we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\Rightarrow \quad (1 + x^2)\frac{dy}{dx} = y$$

Again differentiating w.r.t. x, we get

$$(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{dy}{dx}$$

$$\Rightarrow \quad (1 + x^2)\frac{d^2y}{dx^2} + (2x - 1)\frac{dy}{dx} = 0$$

16. Find: $\displaystyle\int \frac{e^x}{(2 + e^x)(4 + e^{2x})}dx$

Sol. Consider $\displaystyle\int \frac{e^x}{(2 + e^x)(4 + e^{2x})}dx = \int \frac{1}{(2 + t)(4 + t^2)}dt$ | Let $e^x = t$ $\Rightarrow e^x\, dx = dt$

Let $$\frac{1}{(2 + t)(4 + t^2)} = \frac{A}{2 + t} + \frac{Bt + C}{4 + t^2} \quad \text{...}(i)$$

$$\Rightarrow \quad 1 = A(4 + t^2) + (Bt + C)(2 + t)$$

$$\Rightarrow \quad 1 = 4A + At^2 + 2Bt + 2C + Bt^2 + Ct$$

$$\Rightarrow \quad 1 = t^2(A + B) + t(2B + C) + (4A + 2C)$$

Comparing the coefficients, we get

$A + B = 0,\quad 2B + C = 0$ and $4A + 2C = 1$

$\Rightarrow \quad A = -B, C = -2B \quad \Rightarrow \quad -4B - 4B = 1 \quad \Rightarrow \quad B = -\frac{1}{8}$

$\therefore \quad A = \frac{1}{8},\quad B = -\frac{1}{8}$ and $C = \frac{1}{4}$

Substituting in (i) and integrating, we get

$$\int \frac{1}{(2 + t)(4 + t^2)}dt = \frac{1}{8}\int \frac{1}{2 + t}dt + \int \frac{-\frac{1}{8}t + \frac{1}{4}}{4 + t^2}dt$$

$$= \frac{1}{8}\int \frac{1}{2 + t}dt - \frac{1}{16}\int \frac{2t}{4 + t^2}dt + \frac{1}{4}\int \frac{1}{4 + t^2}dt$$

$$= \frac{1}{8}\log|2 + t| - \frac{1}{16}\log|4 + t^2| + \frac{1}{4}\cdot\frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$$\Rightarrow \quad \int \frac{e^x}{(2 + e^x)(4 + e^{2x})}dx = \frac{1}{8}\log|2 + e^x| - \frac{1}{16}\log|4 + e^{2x}| + \frac{1}{8}\tan^{-1}\left(\frac{e^x}{2}\right) + C$$

17. Evaluate: $\int_{-2}^{1} |x^3 - x|\, dx$

Or

Find: $\int e^{2x} \sin(3x + 1)\, dx$

Sol. Consider $\int_{-2}^{1} |x^3 - x|\, dx$

Now $\qquad x^3 - x = x(x^2 - 1) = x(x-1)(x+1) \qquad \ldots(i)$

Zeros are – 1, 0, 1

For $\quad -2 < x < -1, \quad x^3 - x < 0 \qquad$ [from (*i*)]

For $\quad -1 < x < 0, \quad x^3 - x > 0 \qquad$ [from (*i*)]

For $\quad 0 < x < 1, \quad x^3 - x < 0 \qquad$ [from (*i*)]

$$\therefore \int_{-2}^{1} |x^3 - x|\, dx = \int_{-2}^{-1} -(x^3 - x)\, dx + \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} -(x^3 - x)\, dx$$

$$= \left[-\frac{x^4}{4} + \frac{x^2}{2}\right]_{-2}^{-1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[-\frac{x^4}{4} + \frac{x^2}{2}\right]_{0}^{1}$$

$$= \left(-\frac{1}{4} + \frac{1}{2}\right) - (-4 + 2) + (0) - \left(\frac{1}{4} - \frac{1}{2}\right) + \left(-\frac{1}{4} + \frac{1}{2}\right) - 0$$

$$= \frac{1}{4} + 2 + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} + 2 = \frac{11}{4}$$

Or

Consider
$$I = \int \underset{①}{e^{2x}} \sin(\underset{②}{3x} + 1)\, dx$$

$$= e^{2x} \int \sin(3x + 1)\, dx - \int \left\{\frac{d}{dx}(e^{2x}) \cdot \int \sin(3x + 1)\, dx\right\} dx$$

$$= e^{2x} \cdot \left\{\frac{-\cos(3x + 1)}{3}\right\} - \int 2e^{2x} \cdot \left\{\frac{-\cos(3x + 1)}{3}\right\} dx$$

$$= -\frac{1}{3} e^{2x} \cos(3x + 1) + \frac{2}{3} \int \underset{①}{e^{2x}} \cdot \cos(\underset{②}{3x} + 1)\, dx$$

$$= -\frac{1}{3} e^{2x} \cos(3x + 1) + \frac{2}{3}\left[e^{2x} \cdot \frac{\sin(3x + 1)}{3} - \int 2e^{2x} \cdot \frac{\sin(3x + 1)}{3} dx\right]$$

$$\Rightarrow \qquad I = -\frac{1}{3}e^{2x}\cos(3x + 1) + \frac{2}{9}e^{2x}\sin(3x + 1) - \frac{4}{9}I$$

$$\Rightarrow \qquad I + \frac{4}{9}I = \frac{e^{2x}}{9}[-3\cos(3x + 1) + 2\sin(3x + 1)]$$

$$\Rightarrow \qquad \frac{13}{9}I = \frac{e^{2x}}{9}[-3\cos(3x + 1) + 2\sin(3x + 1)]$$

$$\Rightarrow \qquad I = \frac{e^{2x}}{13}[-3\cos(3x + 1) + 2\sin(3x + 1)] + C$$

18. Find the particular solution of the differential equation $2y\, e^{x/y}\, dx + (y - 2x\, e^{x/y})\, dy = 0$, given that $x = 0$ when $y = 1$.

Sol. Consider equation $2y.e^{x/y}\, dx + (y - 2xe^{x/y})dy = 0$

$$\Rightarrow \qquad \frac{dx}{dy} = -\frac{y - 2xe^{x/y}}{2ye^{x/y}} \qquad \ldots(i)$$

Let $x = vy \Rightarrow \dfrac{dx}{dy} = v + y.\dfrac{dv}{dy}$

Substituting in (i), we get

$$v + y\frac{dv}{dy} = -\left(\frac{y - 2vye^{v}}{2ye^{v}}\right) = \frac{-1 + 2ve^{v}}{2e^{v}}$$

$$\Rightarrow \qquad y\frac{dv}{dy} = \frac{-1 + 2ve^{v}}{2e^{v}} - v = \frac{-1 + 2ve^{v} - 2ve^{v}}{2e^{v}} = \frac{-1}{2e^{v}}$$

$$\Rightarrow \qquad e^{v}dv = -\frac{1}{2}\frac{dy}{y} \quad \Rightarrow \quad \int e^{v}dv = -\frac{1}{2}\int\frac{dy}{y}$$

$$\Rightarrow \qquad e^{v} = -\frac{1}{2}\log|y| + C$$

$$\Rightarrow \qquad e^{x/y} = -\frac{1}{2}\log|y| + C \qquad \ldots(ii)$$

Given $x = 0$, when $y = 1$

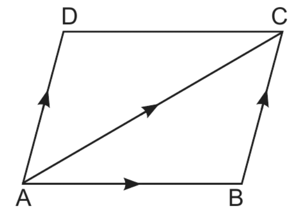
$$\Rightarrow \qquad e^{0} = -\frac{1}{2}\log 1 + C \quad \Rightarrow \quad C = 1$$

Substituting in (ii), we get

$$e^{x/y} = -\frac{1}{2}\log|y| + 1$$ as the particular solution.

19. Find the area of a parallelogram *ABCD* whose side *AB* and the diagonal *AC* are given by the vectors $3\hat{i} + \hat{j} + 4\hat{k}$ and $4\hat{i} + 5\hat{k}$ respectively.

Sol. Given $\overrightarrow{AB} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\overrightarrow{AC} = 4\hat{i} + 5\hat{k}$



We know $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\therefore \quad \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

$= 4\hat{i} + 5\hat{k} - 3\hat{i} - \hat{j} - 4\hat{k} = \hat{i} - \hat{j} + \hat{k}$

$\therefore \quad \overrightarrow{AD} = \overrightarrow{BC} = \hat{i} - \hat{j} + \hat{k}$

$\therefore$ Area of parallelogram $= |\overrightarrow{AB} \times \overrightarrow{AD}|$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}(5) - \hat{j}(-1) + \hat{k}(-4) = 5\hat{i} + \hat{j} - 4\hat{k}$$

$\therefore$ Area of parallelogram $= |5\hat{i} + \hat{j} - 4\hat{k}| = \sqrt{25 + 1 + 16} = \sqrt{42}$ sq units

20. If $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}, \vec{b} = 4\hat{i} - 7\hat{j} + \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 6$.

Sol. Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$ ...(*i*)

$$\therefore \quad \vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ x & y & z \end{vmatrix} = \hat{i}(z + y) - \hat{j}(2z + x) + \hat{k}(2y - x)$$

As $\vec{a} \times \vec{c} = \vec{b} \Rightarrow (y + z)\hat{i} - (2z + x)\hat{j} + (2y - x)\hat{k} = 4\hat{i} - 7\hat{j} + \hat{k}$

$\therefore \quad y + z = 4$ ...(*ii*), $2z + x = 7$ ...(*iii*), and $2y - x = 1$ ...(*iv*)

Also $\vec{a} \cdot \vec{c} = 6 \Rightarrow (2\hat{i} + \hat{j} - \hat{k}).(x\hat{i} + y\hat{j} + z\hat{k}) = 6 \Rightarrow 2x + y - z = 6$ ...(*v*)

From (*ii*) and (*v*), we get

$2x + 2y = 10 \Rightarrow x + y = 5$

Also $2y - x = 1$ [from (*iv*)]

$\therefore \quad 3y = 6 \Rightarrow y = 2, x = 3$ and $z = 2$

$\therefore$ vector $\vec{c} = 3\hat{i} + 2\hat{j} + 2\hat{k}$

21. There are 4 cards numbered 1 to 4, one number on one card. Two cards are drawn at random without replacement. Let *X* denote the sum of the numbers on the two drawn cards. Find the mean and variance of *X*.

Sol. Cards are numbered 1 to 4. When two cards are drawn there can be 6 combinations. i.e. (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4) and possible totals are 3, 4, 5, 6 and 7.

$\therefore$ X : sum of numbers on two cards can take values 3, 4, 5, 6 and 7.

Table for mean and variance of X is as follows:

X	$P(X)$	$X\cdot P(X)$	$X^2\cdot P(X)$
3	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{9}{6}$
4	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{16}{6}$
5	$\frac{2}{6}$	$\frac{10}{6}$	$\frac{50}{6}$
6	$\frac{1}{6}$	$\frac{6}{6}$	$\frac{36}{6}$
7	$\frac{1}{6}$	$\frac{7}{6}$	$\frac{49}{6}$
		$\Sigma X\cdot P(X) = \frac{30}{6} = 5$	$\Sigma X^2\cdot P(X) = \frac{160}{6} = \frac{80}{3}$

$\therefore$ Mean $= \Sigma X\cdot P(X) = 5$

$$\text{Variance} = \Sigma X^2\cdot P(X) - [\Sigma X\cdot P(X)]^2 = \frac{80}{3} - (5)^2 = \frac{80-75}{3} = \frac{5}{3}.$$

22. In a shop X, 30 tins of pure ghee and 40 tins of adulterated ghee which look alike, are kept for sale while in shop Y, similar 50 tins of pure ghee and 60 tins of adulterated ghee are there. One tin of ghee is purchased from one of the randomly selected shops and is found to be adulterated. Find the probability that it is purchased from shop Y. What measures should be taken to stop adulteration?

Sol.

	Pure ghee	**Adulterated ghee**
Shop X:	30	40
Shop Y:	50	60

$$P(X) = \frac{1}{2}, P(Y) = \frac{1}{2}$$

E : tin purchased is found to be adulterated.

$$\therefore \quad P(E/X) = \frac{40}{70} = \frac{4}{7}, \quad P(E/Y) = \frac{60}{110} = \frac{6}{11}$$

Using Bayes' Theorem

Probability that adulterated tin was purchased from shop Y is

$$P(Y/E) = \frac{P(Y)\cdot P(E/Y)}{P(X)\cdot P(E/X) + P(Y)\cdot P(E/Y)} = \frac{\frac{1}{2}\cdot\frac{6}{11}}{\frac{1}{2}\cdot\frac{4}{7}+\frac{1}{2}\cdot\frac{6}{11}} = \frac{\frac{6}{11}}{\frac{4}{7}+\frac{6}{11}}$$

$$= \frac{42}{44+42} = \frac{42}{86} = \frac{21}{43}$$

Food products should be purchased from reliable shop, we should not go in for the cheap product of the same brand, also we must check the information provided on the box.

23. Solve the following *LPP* graphically:

Maximise $Z = 1000x + 600y$

subject to the constraints

$$x + y \le 200$$
$$x \ge 20$$
$$y - 4x \ge 0$$
$$x, y \ge 0.$$

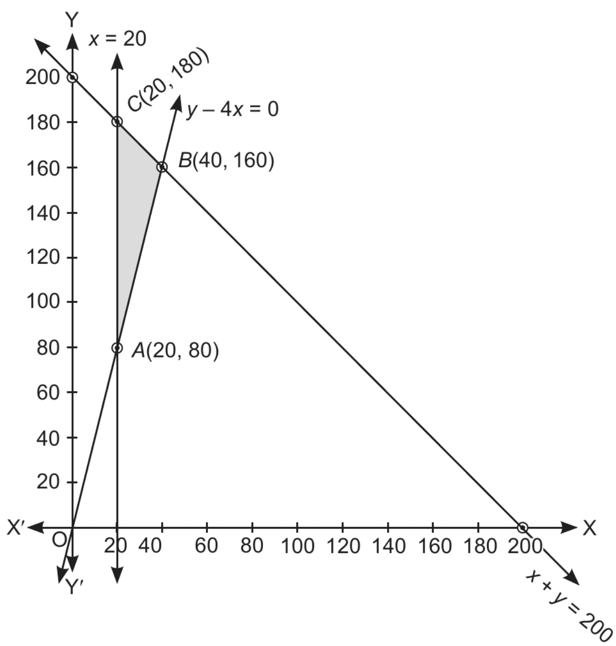
Sol. To maximise $Z = 1000x + 600y$

subject to the constraints

$$x + y \le 200$$
$$x \ge 20$$
$$y - 4x \ge 0 \Rightarrow y \ge 4x$$
$$x, y \ge 0$$

Plotting the graph of the inequations we notice shaded portion is feasible solution.

Possible points for maximum Z are $A(20, 80)$, $B(40, 160)$, $C(20, 180)$



Points	$Z = 1000x + 600y$	Value
$A(20, 80)$	20000 + 48000	68000
$B(40, 160)$	40000 + 96000	1,36,000 ← Maximum
$C(20, 180)$	20000 + 108000	1,28,000

Z is maximum at $B(40, 160)$, i.e. for $x = 40$ and $y = 160$.

24. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ **are two square matrices, find** AB **and hence solve the system of linear equations** $x - y = 3$, $2x + 3y + 4z = 17$ **and** $y + 2z = 7$.

Sol. Consider $AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2+4+0 & 2-2-0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2-2 & 0-4+10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad AB = 6I \quad$ ...(*i*)

From (*i*), we have $AB = 6I \Rightarrow A^{-1}(AB) = 6A^{-1}I \Rightarrow (A^{-1}A)B = 6A^{-1}$

$\Rightarrow \quad IB = 6A^{-1} \Rightarrow A^{-1} = \frac{1}{6}B$

Given equations are

$$x - y = 3$$
$$2x + 3y + 4z = 17$$
$$y + 2z = 7$$

Matrix equation is $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$

$\Rightarrow \quad AX = C \Rightarrow X = A^{-1}C$

$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6+34-28 \\ -12+34-28 \\ 6-17+35 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

$\Rightarrow \quad x = 2, y = -1$ and $z = 4$.

25. Let $A = R - \{3\}, B = R - \{1\}$. Let $f : A \to B$ be defined by $f(x) = \frac{x-2}{x-3}, \forall x \in A$. Show that f is bijective. Also, find

(i) x, if $f^{-1}(x) = 4$

(ii) $f^{-1}(7)$

Or

Let $A = R \times R$ and let * be a binary operation on A defined by $(a, b) * (c, d) = (ad + bc, bd)$ for all $(a, b), (c, d) \in R \times R$.

(i) Show that * is commutative on A.

(ii) Show that * is associative on A.

(iii) Find the identity element of * in A.

Sol. Given, $A = R - \{3\}, B = R - \{1\}$ and $f(x) = \frac{x-2}{x-3}$

For one-one: Let for $x_1, x_2 \in$ A,

$$f(x_1) = f(x_2)$$

$$\Rightarrow \quad \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$$\Rightarrow \quad x_1x_2 - 2x_2 - 3x_1 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6 \quad \Rightarrow \quad x_2 = x_1$$

as $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Hence, function is one-one.

For onto: Let for $y \in B$, there exists $x \in A$ such that $y = f(x) \Rightarrow y = \frac{x-2}{x-3}$

$$\Rightarrow \quad xy - 3y = x - 2$$

$$\Rightarrow \quad xy - x = 3y - 2$$

$$\Rightarrow \quad x(y-1) = 3y - 2$$

$$\Rightarrow \quad x = \frac{3y-2}{y-1} \in A.$$

As function is one-one and onto. Hence bijective.

Given $$f(x) = \frac{x-2}{x-3}$$

Let $y = f(x) \Rightarrow x = f^{-1}(y)$

$$y = \frac{x-2}{x-3} \Rightarrow xy - 3y = x - 2$$

$$\Rightarrow \quad xy - x = 3y - 2 \Rightarrow x(y-1) = 3y - 2$$

$\Rightarrow \qquad x = \dfrac{3y-2}{y-1} \Rightarrow f^{-1}(y) = \dfrac{3y-2}{y-1}$

i.e. $\qquad f^{-1}(x) = \dfrac{3x-2}{x-1}$

(*i*) When $f^{-1}(x) = 4 \Rightarrow \dfrac{3x-2}{x-1} = 4 \Rightarrow 3x-2 = 4x-4 \Rightarrow x = 2$

(*ii*) $f^{-1}(7) = \dfrac{21-2}{7-1} = \dfrac{19}{6}$

Or

Given $(a, b)*(c, d) = (ad + bc, bd)$ for $(a, b), (c, d) \in A = R \times R$.

(*i*) **For commutative:** Let $(a, b), (c, d) \in A$

$$(a, b) * (c, d) = (ad + bc, bd)$$

and $$(c, d) * (a, b) = (cb + da, db)$$

As $ad + bc = cb + da$ and $bd = db$ for $a, b, c, d \in R$

Hence, $(a, b) * (c, d) = (c, d) * (a, b)$

Hence, commutative.

(*ii*) **For Associative:** Let $(a, b), (c, d), (e, f) \in A$

$$\begin{aligned}\{(a, b)*(c, d)\}*(e, f) &= (ad + bc, bd)*(e, f)\\ &= ((ad + bc)f + bde, bdf)\\ &= (adf + bcf + bde, bdf)\end{aligned}$$

and $$\begin{aligned}(a, b)*\{(c, d)*(e, f)\} &= (a, b) * (cf + de, df)\\ &= (adf + b(cf + de), bdf)\\ &= (adf + bcf + bde, bdf)\end{aligned}$$

We notice $\{(a, b)*(c, d)\}*(e, f) = (a, b) * \{(c, d)*(e, f)\}$

Hence, associative.

(*iii*) **For identity:** Let $(e_1, e_2) \in A$ be identity element.

$$\therefore \quad (a, b) * (e_1, e_2) = (e_1, e_2) * (a, b) = (a, b) \text{ for } (a, b) \in A$$

$$\Rightarrow \quad (ae_2 + be_1, be_2) = (e_1b + e_2a, e_2b) = (a, b)$$

$$\Rightarrow \quad ae_2 + be_1 = a \text{ and } be_2 = b$$

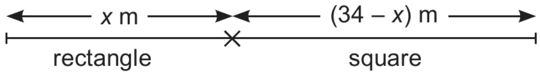
$$\Rightarrow \quad e_2 = 1$$

$$\Rightarrow \quad a + be_1 = a \Rightarrow be_1 = 0 \Rightarrow e_1 = 0$$

$\therefore \quad (0, 1) \in A$ is identity.

26. A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a rectangle whose length is twice its breadth. What should be the lengths of the two pieces, so that the combined area of the square and the rectangle is minimum?

Sol.



Let wire is cut at x m and made into a rectangle, whose breadth is a and length $2a$.
For rectangle,

$$\therefore \quad 2(a + 2a) = x \Rightarrow 6a = x \Rightarrow a = \frac{x}{6} \text{ m}$$

$$\therefore \quad \text{Area of rectangle} = a(2a) = 2a^2 = 2\left(\frac{x}{6}\right)^2 = \frac{x^2}{18}$$

$$\text{For square,} \quad 4 \times \text{side} = (34 - x) \Rightarrow \text{side} = \frac{34 - x}{4} \text{m}$$

$$\therefore \quad \text{Area of square} = \frac{1}{16}(34 - x)^2$$

$$\text{Combined area } (A) = \frac{x^2}{18} + \frac{1}{16}(34 - x)^2$$

$$\frac{dA}{dx} = \frac{2x}{18} + \frac{2}{16}(34 - x)\cdot(-1) = \frac{x}{9} - \left(\frac{34 - x}{8}\right) = \frac{8x - 306 + 9x}{72}$$

$$\frac{dA}{dx} = \frac{17x - 306}{72}$$

$$\text{For minimum area,} \quad \frac{dA}{dx} = 0$$

$$\Rightarrow \quad 17x - 306 = 0 \Rightarrow x = \frac{306}{17} = 18$$

$$\frac{d^2A}{dx^2} = \frac{17}{72}$$

$$\left.\frac{d^2A}{dx^2}\right]_{x = 18} = \frac{17}{72} > 0$$

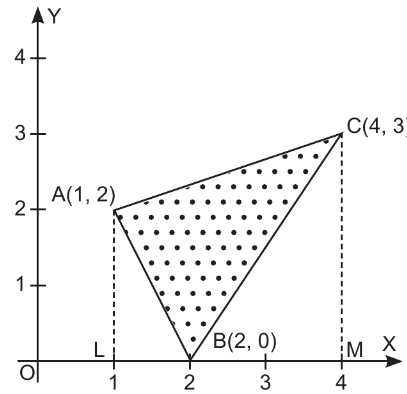
$\therefore$ For $x = 18$, A is minimum. Hence, wire must be cut at 18 m to made into a rectangle and remaining 16 m piece to made into a square for a minimum combined area.

27. Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are $A(1, 2)$, $B(2, 0)$ and $C(4, 3)$.

Or

Using integration, find the area of the region $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$.

Sol. The vertices of a triangle are $A(1, 2)$, $B(2, 0)$ and $C(4, 3)$. Plotting graph, we notice we have to find area of shaded region.



$\text{ar}(ABC) = \text{ar}(LACM) - \text{ar}(\Delta LAB) - \text{ar}(\Delta BCM) \ldots(i)$

Equation of AB: $A(1, 2)$, $B(2, 0)$

$$y - 0 = \frac{0-2}{2-1}(x-2) \Rightarrow y = -2x + 4$$

Equation of BC: $B(2, 0)$, $C(4, 3)$

$$y - 0 = \frac{3-0}{4-2}(x-2) \Rightarrow y = \frac{3}{2}x - 3$$

Equation of AC: $A(1, 2)$, $C(4, 3)$

$$y - 3 = \frac{3-2}{4-1}(x-4) \Rightarrow y - 3 = \frac{1}{3}x - \frac{4}{3} \Rightarrow y = \frac{1}{3}x + \frac{5}{3}$$

$$\therefore \quad \text{ar}(ABC) = \int_1^4 y_{AC}\,dx - \int_1^2 y_{AB}\,dx - \int_2^4 y_{BC}\,dx$$

$$= \int_1^4 \left(\frac{1}{3}x + \frac{5}{3}\right)dx - \int_1^2 (-2x+4)\,dx - \int_2^4 \left(\frac{3}{2}x - 3\right)dx$$

$$= \left[\frac{x^2}{6} + \frac{5x}{3}\right]_1^4 - [-x^2 + 4x]_1^2 - \left[\frac{3x^2}{4} - 3x\right]_2^4$$

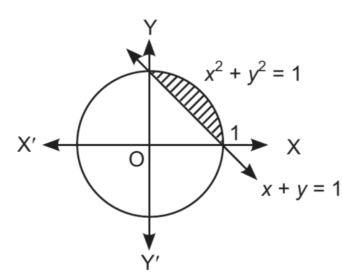
$$= \left(\frac{16}{6} + \frac{20}{3}\right) - \left(\frac{1}{6} + \frac{5}{3}\right) - (-4 + 8) + (-1 + 4) - (12 - 12) + (3 - 6)$$

$$= \frac{15}{6} + \frac{15}{3} - 4 + 3 - 0 - 3$$

$$= \frac{45}{6} - 4 = \frac{45-24}{6} = \frac{21}{6} = \frac{7}{2} \text{ sq units}$$

Or

Curves are $x^2 + y^2 = 1$ and $x + y = 1$



$$\text{Area bounded} = \int_0^1 \{\sqrt{1-x^2} - (1-x)\}\,dx$$

$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x - x + \frac{x^2}{2}\right]_0^1$$

$$= \left(0 + \frac{1}{2}\sin^{-1}1 - 1 + \frac{1}{2}\right) - 0$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2} = \frac{1}{4}(\pi - 2) \text{ sq units}$$

28. Find the particular solution of the differential equation $\frac{dy}{dx} - 3y \cot x = \sin 2x$, given that $y = 2$ when $x = \frac{\pi}{2}$.

Sol. Consider equation $\frac{dy}{dx} - 3y \cot x = \sin 2x$

Here, $P(x) = -3\cot x$ and $Q(x) = \sin 2x$

Integrating factor $(I.F.) = e^{\int -3\cot x dx} = e^{-3\int \cot x dx}$

$$= e^{-3\log|\sin x|} = e^{\log|\sin x|^{-3}} = \frac{1}{\sin^3 x}$$

Solution is $(I.F.)\ y = \int \{(I.F.)Q(x)\}\,dx$

$$\frac{1}{\sin^3 x}\cdot y = \int \frac{1}{\sin^3 x}\cdot \sin 2x\, dx$$

$$= \int \frac{2\sin x \cos x}{\sin^3 x} dx = 2\int \operatorname{cosec} x \cot x\, dx$$

$\Rightarrow$ $\frac{1}{\sin^3 x}\cdot y = -2\operatorname{cosec} x + C$

$\Rightarrow$ $y = -2\sin^2 x + C\sin^3 x$ ...(*i*)

Given $y = 2$, when $x = \frac{\pi}{2}$

$\therefore$ $2 = -2\sin^2\frac{\pi}{2} + C.\sin^3\frac{\pi}{2}$

$\Rightarrow$ $2 = -2 + C \Rightarrow C = 4$

Substituting in (*i*), we get

$y = -2\sin^2 x + 4\sin^3 x$ as the particular solution.

29. Find the vector equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Hence find whether the plane thus obtained contains the line $\frac{x+2}{5} = \frac{y-3}{4} = \frac{z}{5}$ or not.

Or

Find the image P' of the point P having position vector $\hat{i} + 3\hat{j} + 4\hat{k}$ in the plane $\vec{r}\cdot(2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$. Hence find the length of PP'.

Sol. General equation of the plane through the intersection of planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ is

$x + y + z - 1 + \lambda(2x + 3y + 4z - 5) = 0$

$\Rightarrow (1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z + (-1 - 5\lambda) = 0$ ...(*i*)

If plane (*i*) is perpendicular to plane $x - y + z = 0$, then

$1(1 + 2\lambda) - 1(1 + 3\lambda) + 1(1 + 4\lambda) = 0$

$\Rightarrow 1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0 \Rightarrow 3\lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{3}$

Substituting in (*i*), plane is

$\left(1-\frac{2}{3}\right)x + (1-1)y + \left(1-\frac{4}{3}\right)z + \left(-1+\frac{5}{3}\right) = 0$

$\Rightarrow \quad \frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0 \Rightarrow x - z + 2 = 0$ ...(*ii*)

In vector form plane is $\vec{r}\cdot(\hat{i} - \hat{k}) + 2 = 0$

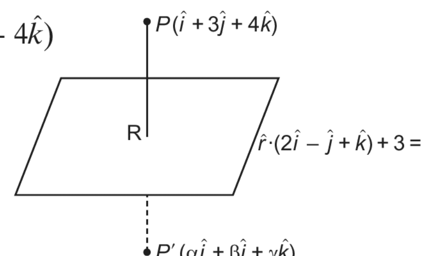
If plane (*ii*), i.e. $x - z + 2 = 0$ contains the line $\frac{x+2}{5} = \frac{y-3}{4} = \frac{z}{5}$, the point $(-2, 3, 0)$ on the line must lie on plane, i.e. $-2 - 0 + 2 = 0$, true.

Also, line must be perpendicular to normal to plane

i.e. $1 \times 5 + 0 \times 4 - 1 \times 5 = 0 \quad \Rightarrow \quad 5 - 5 = 0$,

Hence, the plane contains the line.

Or



Let $P'(\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k})$ be image of point $P(\hat{i} + 3\hat{j} + 4\hat{k})$ in the plane $\vec{r}\cdot(2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$

Let PP' meets plane on R.

Then PP' is perpendicular to the plane and R is mid-point of PP'.

Position vector of R is $\left(\frac{1+\alpha}{2}\hat{i} + \frac{3+\beta}{2}\hat{j} + \frac{4+\gamma}{2}\hat{k}\right)$ ...(*i*)

Also equation of PR is $\vec{r} = (\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$

General point on line PR is $\vec{r} = (1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (4 + \lambda)\hat{k}$ ...(*ii*)

If this point lies on the plane $\vec{r}\cdot(2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$

then, $2(1 + 2\lambda) - 1(3 - \lambda) + 1(4 + \lambda) + 3 = 0$

$\Rightarrow \quad 2 + 4\lambda - 3 + \lambda + 4 + \lambda + 3 = 0 \Rightarrow 6\lambda + 6 = 0 \Rightarrow \lambda = -1$

Substituting in (*ii*) position vector of mid-point R is $R(-\hat{i} + 4\hat{j} + 3\hat{k})$ ...(*iii*)

From (*i*) and (*iii*), we get

$$\left(\frac{1+\alpha}{2}\right)\hat{i} + \left(\frac{3+\beta}{2}\right)\hat{j} + \left(\frac{4+\gamma}{2}\right)\hat{k} = -\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\Rightarrow \frac{1+\alpha}{2} = -1, \frac{3+\beta}{2} = 4, \frac{4+\gamma}{2} = 3$$

$\Rightarrow \alpha = -3, \beta = 5$ and $\gamma = 2$

$\therefore$ Image is (–3, 5, 2)

Position vector of image P' is $(-3\hat{i} + 5\hat{j} + 2\hat{k})$

Also $\overrightarrow{PP'} = (-3\hat{i} + 5\hat{j} + 2\hat{k}) - (\hat{i} + 3\hat{j} + 4\hat{k}) = -4\hat{i} + 2\hat{j} - 2\hat{k}$

$\therefore$ Distance $PP' = |\overrightarrow{PP'}| = \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$ units

[SET II: UNCOMMON QUESTIONS TO SET I]

12. The radius r of the base of a right circular cone is decreasing at the rate of 2 cm/min. and its height h is increasing at the rate of 3 cm/min. When $r = 3.5$ cm and $h = 6$ cm, find the rate of change of the volume of the cone. [Use $\pi = \frac{22}{7}$]

Sol. Given $\frac{dr}{dt} = -2$ cm/min (decreasing) and $\frac{dh}{dt} = 3$cm/min (increasing) ...(*i*)

$$\text{Volume of cone } V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\frac{d}{dt}(r^2h) = \frac{1}{3}\pi\left[r^2 \cdot \frac{dh}{dt} + 2rh\frac{dr}{dt}\right]$$

$$= \frac{1}{3}\pi(3r^2 - 4rh) \quad \text{[using (i)]}$$

$$\therefore \left.\frac{dV}{dt}\right]_{\substack{r = 3.5\text{ cm} \\ h = 6\text{ cm}}} = \frac{1}{3}\pi[3(3.5)^2 - 4 \times 3.5 \times 6]$$

$$= \frac{1}{3}\pi(3 \times 12.25 - 84)$$

$$= \frac{1}{3}\pi(36.75 - 84)$$

$$= \frac{1}{3}\pi \times (-47.25) = -15.75\,\pi \text{ cm}^3/\text{min}$$

Volume is decreasing at the rate of 15.75π cm^3/min.

20. Find: $\int \frac{x\,dx}{(2+x^2)(4+x^4)}$

Sol. Consider $\int \frac{x}{(2+x^2)(4+x^4)}dx = \frac{1}{2}\int \frac{1}{(2+t)(4+t^2)}$ ...(i)

Let $x^2 = t$
$\Rightarrow 2x\,dx = dt$
$\Rightarrow x\,dx = \frac{1}{2}dt$

Let $$\frac{1}{(2+t)(4+t^2)} = \frac{A}{2+t} + \frac{Bt+C}{4+t^2}$$

$\Rightarrow 1 = A(4+t^2) + (Bt+C)(2+t) = 4A + At^2 + 2Bt + 2C + Bt^2 + Ct$

$\Rightarrow 1 = t^2(A+B) + t(2B+C) + (4A+2C)$

Comparing coefficients, we get

$A + B = 0$, $2B + C = 0$ and $4A + 2C = 1$

$\Rightarrow \quad A = -B$, $2B = -C$ and $-4B - 4B = 1 \Rightarrow B = -\frac{1}{8}$

$\therefore \quad A = \frac{1}{8}, C = \frac{1}{4}, B = -\frac{1}{8}$

From (i), $\frac{1}{2}\int\left\{\frac{\frac{1}{8}}{2+t} + \frac{-\frac{1}{8}t + \frac{1}{4}}{4+t^2}\right\}dt$

$$= \frac{1}{16}\int\frac{1}{2+t}dt - \frac{1}{32}\int\frac{2t}{4+t^2}dt + \frac{1}{8}\int\frac{1}{4+t^2}dt$$

$$= \frac{1}{16}\log|2+t| - \frac{1}{32}\log|4+t^2| + \frac{1}{8}\times\frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$$\int\frac{x\,dx}{(2+x^2)(4+x^4)} = \frac{1}{16}\log|2+x^2| - \frac{1}{32}\log|4+x^4| + \frac{1}{16}\tan^{-1}\frac{x^2}{2} + C$$

21. Solve the following *LPP* graphically:

Minimise $Z = 3x + 9y$

subject to the constraints

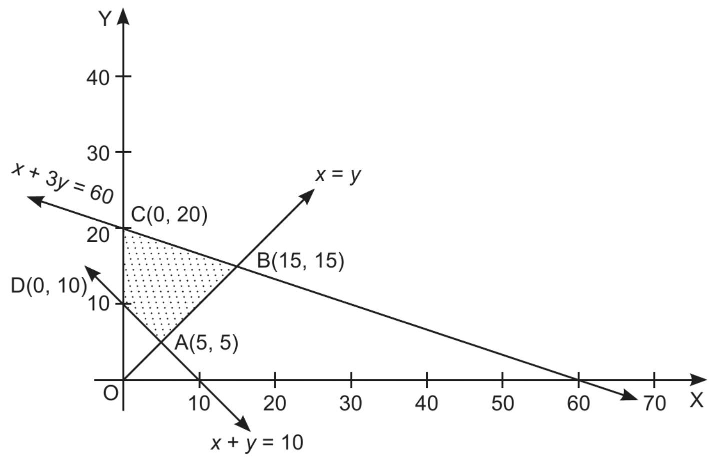
$$x + 3y \le 60$$
$$x + y \ge 10$$
$$x \le y$$
$$x \ge 0,\ y \ge 0.$$

Sol. To minimise $Z = 3x + 9y$

subject to the constraints

$$x \ge 0,\ y \ge 0$$
$$x + 3y \le 60$$
$$x + y \ge 10$$
$$x \le y$$

Plotting the graph of the inequations we notice shaded portion is feasible solution.



Possible points for minimum Z are $A(5, 5)$, $B(15, 15)$, $C(0, 20)$ and $D(0, 10)$.

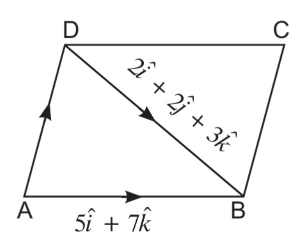
Points	$Z = 3x + 9y$	**Value**	
$A(5, 5)$	$15 + 45$	60	← Minimum
$B(15, 15)$	$45 + 135$	180	
$C(0, 20)$	$0 + 180$	180	
$D(0, 10)$	$0 + 90$	90	

Z is minimum for $A(5, 5)$, i.e. $x = 5, y = 5$

22. Find the area of a parallelogram *ABCD* whose side *AB* and the diagonal *DB* are given by the vectors $5\hat{i} + 7\hat{k}$ and $2\hat{i} + 2\hat{j} + 3\hat{k}$ respectively.

Sol. Given for parallelogram $ABCD$

$$\overrightarrow{AB} = 5\hat{i} + 7\check{k} \text{ and } \overrightarrow{DB} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$



We have $\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$

$\Rightarrow$ $\overrightarrow{AD} = \overrightarrow{AB} - \overrightarrow{DB}$

$= 5\hat{i} + 7\hat{k} - 2\hat{i} - 2\hat{j} - 3\hat{k}$

$= 3\hat{i} - 2\hat{j} + 4\hat{k}$

$\therefore$ Area of parallelogram $= |\overrightarrow{AB} \times \overrightarrow{AD}|$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 7 \\ 3 & -2 & 4 \end{vmatrix}$$

$$= \hat{i}(14) - \hat{j}(-1) + \hat{k}(-10) = 14\hat{i} + \hat{j} - 10\hat{k}$$

$\therefore$ Area of parallelogram $= |14\breve{i} + \hat{j} - 10\hat{k}| = \sqrt{196 + 1 + 100} = \sqrt{297}$ sq units

23. Prove that: $\tan^{-1} 2x + \tan^{-1} \dfrac{4x}{1 - 4x^2} = \tan^{-1}\left(\dfrac{6x - 8x^3}{1 - 12x^2}\right)$; $|x| < \dfrac{1}{2\sqrt{3}}$

Sol. LHS $= \tan^{-1} 2x + \tan^{-1} \dfrac{4x}{1 - 4x^2} = \tan^{-1}\left[\dfrac{2x + \dfrac{4x}{1 - 4x^2}}{1 - (2x)\left(\dfrac{4x}{1 - 4x^2}\right)}\right]$

$$= \tan^{-1}\left[\frac{2x - 8x^3 + 4x}{1 - 4x^2} \times \frac{1 - 4x^2}{1 - 4x^2 - 8x^2}\right]$$

$$= \tan^{-1}\left[\frac{6x - 8x^3}{1 - 12x^2}\right] = \text{RHS}$$

28. If $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$ find A^{-1} and hence solve the system of equations $x + 2y + 5z = 10$, $x - y - z = -2$ and $2x + 3y - z = -11$.

Sol. Consider $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$

We have $A^{-1} = \dfrac{1}{|A|} adj.A$ ...(*i*)

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{vmatrix}$$

$$= 1(1 + 3) - 2(-1+2) + 5(3+2) = 4 - 2 + 25 = 27 \neq 0$$

Hence, A^{-1} exists.

Now, $A_{11} = 4$, $A_{12} = -1$, $A_{13} = 5$

$A_{21} = 17$, $A_{22} = -11$, $A_{23} = 1$

$A_{31} = 3$, $A_{32} = 6$, $A_{33} = -3$

Matrix formed by cofactors of each element in $A = \begin{bmatrix} 4 & -1 & 5 \\ 17 & -11 & 1 \\ 3 & 6 & -3 \end{bmatrix}$

$$\therefore \quad Adj\,A = \begin{bmatrix} 4 & -1 & 5 \\ 17 & -11 & 1 \\ 3 & 6 & -3 \end{bmatrix}' = \begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$$

$\therefore$ From (*i*), $$A^{-1} = \frac{1}{27}\begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix} \quad ...(ii)$$

Consider equations

$$x + 2y + 5z = 10$$

$$x - y - z = -2$$

and $$2x + 3y - z = -11$$

Corresponding matrix equation is

$$\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ -11 \end{bmatrix}$$

i.e. $AX = B$ is matrix equation, Its solution is $X = A^{-1}B$

$$\Rightarrow \quad X = \frac{1}{27}\begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}\begin{bmatrix} 10 \\ -2 \\ -11 \end{bmatrix} \quad \text{[from } (ii)\text{]}$$

$$= \frac{1}{27}\begin{bmatrix} 40 - 34 - 33 \\ -10 + 22 - 66 \\ 50 - 2 + 33 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} -27 \\ -54 \\ 81 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad x = -1, y = -2, z = 3$ is solution

29. Find the particular solution of the differential equation $dy = \cos x\,(2 - y \operatorname{cosec} x)\,dx$, given that $y = 2$ when $x = \dfrac{\pi}{2}$.

Sol. Consider equation $$dy = \cos x(2 - y \operatorname{cosec} x)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\cos x - y\cot x$$

$$\Rightarrow \quad \frac{dy}{dx} + \cot x \cdot y = 2\cos x$$

Here $P(x) = \cot x$, $Q(x) = 2\cos x$

Integrating factor $(I.F.) = e^{\int \cot x\, dx} = e^{\log|\sin x|} = \sin x$

Its solution is $(I.F.)\, y = \int \{(I.F).Q(x)\}dx$

$$\Rightarrow \quad \sin x.y = \int \sin x.(2\cos x)\,dx = \int \sin 2x\, dx$$

$$\Rightarrow \quad \sin x.y = -\frac{\cos 2x}{2} + C \qquad ...(i)$$

Given $y = 2$, when $x = \frac{\pi}{2}$

$$\Rightarrow \quad \sin\frac{\pi}{2}\cdot 2 = -\frac{\cos\pi}{2} + C$$

$$\Rightarrow \quad 2 = \frac{1}{2} + C \Rightarrow C = \frac{3}{2}$$

Substituting in (i), we get

$$\sin x.y = -\frac{1}{2}\cos 2x + \frac{3}{2} \text{ as particular solution}$$

[SET III: UNCOMMON QUESTIONS TO SET I & II]

12. The radius r of a right circular cylinder is increasing uniformly at the rate of 0.3 cm/s and its height h is decreasing at the rate of 0.4 cm/s. When $r = 3.5$ cm and $h = 7$ cm, find the rate of change of the curved surface area of the cylinder. [Use $\pi = \frac{22}{7}$]

Sol. $\frac{dr}{dt} = 0.3$ cm/s (increasing) and $\frac{dh}{dt} = -0.4$ cm/s (decreasing)

Curved surface area $C = 2\pi rh$

$$\frac{dC}{dt} = 2\pi\frac{d}{dt}(rh) = 2\pi\left(r\cdot\frac{dh}{dt} + h\frac{dr}{dt}\right)$$

$$= 2\pi(-0.4r + 0.3h)$$

$$\therefore \left.\frac{dC}{dt}\right]_{\substack{r = 3.5\text{ cm} \\ h = 7\text{ cm}}} = 2\pi(-0.4 \times 3.5 + 0.3 \times 7) = 2\pi(-1.4 + 2.1) = 1.4\pi \text{ cm}^2/\text{s} = 4.4 \text{ cm}^2/\text{s}$$

$\therefore$ Curved surface area is increasing at the rate of 4.4 cm^2/s

20. Solve the following *LPP* graphically:

Maximise $Z = 105x + 90y$

subject to the constraints

$$x + y \le 50$$

$$2x + y \le 80$$

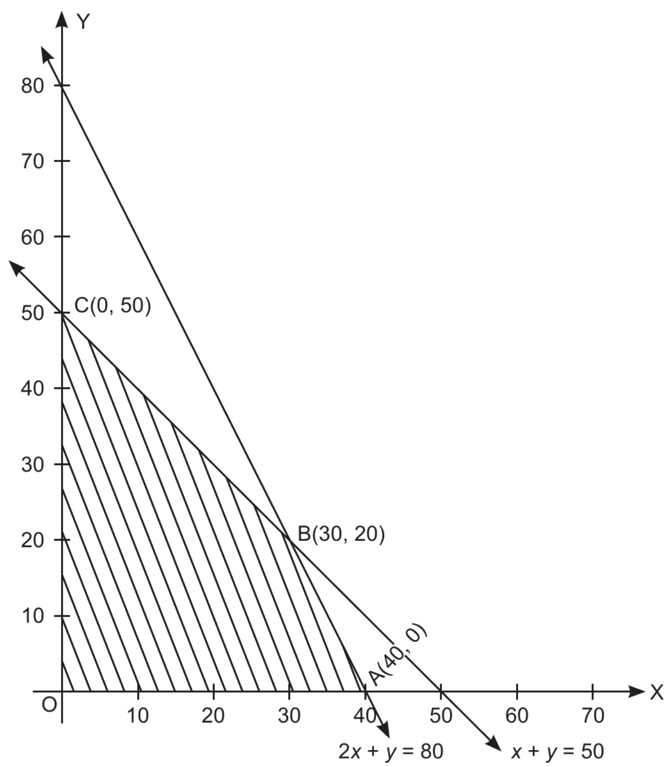
$$x \ge 0,\ y \ge 0.$$

Sol. To maximise $Z = 105x + 90y$

subject to the constraints

$$x \geq 0, \quad y \geq 0$$
$$x + y \leq 50$$
$$2x + y \leq 80$$

Plotting the graph of inequations we notice shaded portion is feasible solution. Possible points for maximum Z are $A(40, 0)$, $B(30, 20)$ and $C(0, 50)$.



Points	$Z = 105x + 90y$	Value	
$A(40, 0)$	4200 + 0	4200	
$B(30, 20)$	3150 + 1800	4950	← Maximum
$C(0, 50)$	0 + 4500	4500	

Z is maximum for $B(30, 20)$, i.e. $x = 30$ and $y = 20$.

21. Find the general solution of the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x.$

Sol. Consider equation,

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)} \quad ...(i)$$

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Substituting in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx\cos v + x}{x\cos v} = \frac{v\cos v + 1}{\cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v = \frac{v\cos v + 1 - v\cos v}{\cos v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1}{\cos v}$$

$$\Rightarrow \quad \cos v\, dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \cos v\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \sin v = \log|x| + C$$

$$\Rightarrow \quad \sin\left(\frac{y}{x}\right) = \log|x| + C \text{ is general solution.}$$

22. Prove that: $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2; -1 < x < 1$

Sol. Let $x^2 = \cos\theta \Rightarrow \theta = \cos^{-1}x^2 \quad ...(i)$

$$\text{LHS} = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}\right)$$

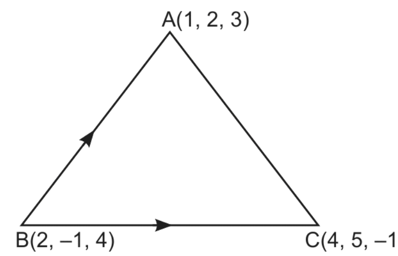
$$= \tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right) \quad \text{[On dividing numerator and denominator by } \sqrt{2}\cos\frac{\theta}{2}\text{]}$$

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

$$= \frac{\pi}{4}+\frac{\theta}{2} = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 = \text{RHS} \qquad \text{[from } (i)\text{]}$$

23. Using vectors, find the area of triangle *ABC*, with vertices *A*(1, 2, 3), *B*(2, –1, 4) and *C*(4, 5, –1).

Sol. Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}|$



$$\overrightarrow{BC} = (4\hat{i}+5\hat{j}-\hat{k})-(2\hat{i}-\hat{j}+4\hat{k})$$

$$= 2\hat{i}+6\hat{j}-5\hat{k}$$

and $\overrightarrow{BA} = (\hat{i}+2\hat{j}+3\hat{k})-(2\hat{i}-\hat{j}+4\hat{k})$

$$= -\hat{i}+3\hat{j}-\hat{k}$$

$$\overrightarrow{BC} \times \overrightarrow{BA} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & -5 \\ -1 & 3 & -1 \end{vmatrix} = \hat{i}(9)-\hat{j}(-7)+\hat{k}(12) = 9\hat{i}+7\hat{j}+12\hat{k}$$

$\therefore$ Area of triangle $= \frac{1}{2}|9\hat{i}+7\hat{j}+12\hat{k}| = \frac{1}{2}\sqrt{81+49+144} = \frac{1}{2}\sqrt{274}$ sq units

28. If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$, find A^{-1} and hence solve the system of equations $x - 2y = 10$, $2x + y + 3z = 8$ and $-2y + z = 7$.

Sol. Consider $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}\text{adj}\,A \qquad ...(i)$$

$$|A| = \begin{vmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{vmatrix} = 1(7)+2(2)+0 = 11 \neq 0$$

Hence, A^{-1} exists.

Now, $A_{11} = 7, \quad A_{12} = -2, \quad A_{13} = -4$

$A_{21} = 2, \quad A_{22} = 1, \quad A_{23} = 2$

$A_{31} = -6, \quad A_{32} = -3, \quad A_{33} = 5$

Matrix formed by cofactors of each element in A

$$= \begin{bmatrix} 7 & -2 & -4 \\ 2 & 1 & 2 \\ -6 & -3 & 5 \end{bmatrix}$$

$$\therefore \quad \text{Adj}\, A = \begin{bmatrix} 7 & -2 & -4 \\ 2 & 1 & 2 \\ -6 & -3 & 5 \end{bmatrix}' = \begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$\therefore \text{ From } (i), \quad A^{-1} = \frac{1}{11}\begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$$

Consider equations

$$x - 2y = 10$$
$$2x + y + 3z = 8$$
$$-2y + z = 7$$

Corresponding matrix equation is

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

i.e. $AX = B$ is matrix equation. Its solution is $X = A^{-1}B$

$$\Rightarrow \quad X = \frac{1}{11}\begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}\begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} 70 + 16 - 42 \\ -20 + 8 - 21 \\ -40 + 16 + 35 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 44 \\ -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad x = 4, y = -3$ and $z = 1$ is solution.

29. Find the particular solution of the differential equation

$$(1 + y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0, \text{ given that } y = 0 \text{ when } x = 1.$$

Sol. Consider equation $(1 + y^2) + \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = 0$

$$\Rightarrow \quad (x - e^{\tan^{-1}y})\frac{dy}{dx} = -(1 + y^2)$$

$$\Rightarrow \quad \frac{-(x - e^{\tan^{-1}y})}{1+y^2} = \frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{x}{1+y^2} + \frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{1+y^2}\cdot x = \frac{e^{\tan^{-1}y}}{1+y^2}$$

Here, $P(y) = \frac{1}{1+y^2}, \quad Q(y) = \frac{e^{\tan^{-1}y}}{1+y^2}$

$$I.F. = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Solution is $\quad (I.F.)x = \int \{(I.F.)\cdot Q(y)\}\, dy$

$$e^{\tan^{-1}y}x = \int e^{\tan^{-1}y}\cdot\frac{e^{\tan^{-1}y}}{1+y^2}\,dy = \int \frac{e^{2\tan^{-1}y}}{1+y^2}dy$$

Let $\tan^{-1}y = t$

$\Rightarrow \frac{1}{1+y^2}dy = dt$

$$e^{\tan^{-1}y}x = \int e^{2t}dt$$

$$= \frac{e^{2t}}{2} + C$$

$$e^{\tan^{-1}y}x = \frac{1}{2}e^{2\tan^{-1}y} + C \qquad ...(i)$$

Given $y = 0$ when $x = 1$

$$\therefore \quad e^{\tan^{-1}0}\cdot 1 = \frac{1}{2}e^{2\tan^{-1}0} + C$$

$$\Rightarrow \quad 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

Substituting in (i), we get

$$e^{\tan^{-1}y}\cdot x = \frac{1}{2}e^{2\tan^{-1}y} + \frac{1}{2}$$

$$\Rightarrow \quad x = \frac{1}{2}e^{\tan^{-1}y} + \frac{1}{2}e^{-\tan^{-1}y} \text{ is particular solution.}$$