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Examination Papers, 2020

[Delhi Set-I, II, III]

Time allowed: 3 Hours] [Maximum Marks: 80

General Instructions:

Read the following instructions very carefully and strictly follow them:

(i) This question paper comprises **four** sections – **A, B, C** and **D**.
This question paper carries **36** questions. All questions are compulsory.

(ii) **Section A** – Question no. **1** to **20** comprises of **20** questions of **one** mark each.

(iii) **Section B** – Question no. **21** to **26** comprises of **6** questions of **two** marks each.

(iv) **Section C** – Question no. **27** to **32** comprises of **6** questions of **four** marks each.

(v) **Section D** – Question no. **33** to **36** comprises of **4** questions of **six** marks each.

(vi) There is no overall choice in the question paper. However, an internal choice has been provided in **3** questions of one mark, **2** questions of two marks, **2** questions of four marks and **2** questions of six marks. Only one of the choices in such questions have to be attempted.

(vii) In addition to this, separate instruction are given with each section and question, wherever necessary.

(viii) Use of calculators is not permitted.

SET-I

SECTION – A

Question numbers 1 to 10 are multiple choice questions. Select the correct option:

1. If A is a square matrix of order 3, such that A (adj A) = 10 I, then $|\text{adj } A|$ is equal to

(*a*) 1 **(*b*) 10** **(*c*) 100** **(*d*) 101**

Sol. (*c*), as $A \cdot (\text{Adj } A) = 10I = |A| I \Rightarrow |A| = 10$

Also, $|\text{Adj } A| = |A|^{3-1} = (10)^2 = 100$

2. If A is a 3×3 matrix such that $|A| = 8$, then $|3A|$ equals.

(*a*) 8 **(*b*) 24** **(*c*) 72** **(*d*) 216**

Sol. (*d*), as $|3A| = 3^3 |A| = 27 \times 8 = 216$

3. If $y = Ae^{5x} + Be^{-5x}$, then $\frac{d^2y}{dx^2}$ is equal to

(*a*) 25*y* **(*b*) 5*y*** **(*c*) –25*y*** **(*d*) 15*y***

Sol. (*a*), as $y' = 5Ae^{5x} - 5Be^{-5x}$

and $y'' = 25Ae^{5x} + 25Be^{-5x} = 25y$

4. $\int x^2 e^{x^3} dx$ equals to

(*a*) $\frac{1}{3}e^{x^3} + C$ **(*b*) $\frac{1}{3}e^{x^4} + C$** **(*c*) $\frac{1}{2}e^{x^3} + C$** **(*d*) $\frac{1}{2}e^{x^2} + C$**

Sol. (*a*), $\int x^2 \cdot e^{x^3} dx = \frac{1}{3}\int e^t dt$ | Let $x^3 = t$ $\Rightarrow 3x^2 dx = dt$

$= \frac{1}{3}e^t + C = \frac{1}{3}e^{x^3} + C$

5. If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along three mutually perpendicular directions, then

(*a*) $\hat{i} \cdot \hat{j} = 1$ **(*b*) $\hat{i} \times \hat{j} = 1$** **(*c*) $\hat{i} \cdot \hat{k} = 0$** **(*d*) $\hat{i} \times \hat{k} = 0$**

Sol. (*c*), as $\hat{i} \cdot \hat{k} = 0$

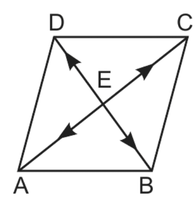
6. *ABCD* is a rhombus whose diagonals intersect at *E*. Then $\overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED}$ equals

(*a*) $\vec{0}$ **(*b*) $\overrightarrow{AD}$** **(*c*) $2\overrightarrow{BC}$** **(*d*) $2\overrightarrow{AD}$**

Sol. (*a*), as $\overrightarrow{EB} = -\overrightarrow{ED}$

and $\overrightarrow{EA} = -\overrightarrow{EC}$

$\Rightarrow \overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED} = \vec{0}$



7. The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{4-z}{k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{-2}$ are mutually perpendicular if the value of *k* is

(*a*) $-\frac{2}{3}$ **(*b*) $\frac{2}{3}$** **(*c*) –2** **(*d*) 2**

Sol. (*a*), as lines are $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$

and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{-2}$

If lines are perpendicular then, $k \times 1 + 2 \times 1 + (-2) \times (-k) = 0$

$\Rightarrow 3k = -2 \Rightarrow k = -\frac{2}{3}$

8. The graph of the inequality $2x + 3y > 6$ is

(*a*) half plane that contains the origin.

(*b*) half plane that neither contains the origin nor the points of the line $2x + 3y = 6$

(*c*) whole XOY – plane excluding the points on the line $2x + 3y = 6$.

(*d*) entire XOY plane.

Sol. (*b*), half plane that neither contains the origin as $0 + 0 > 6$ is false, and only inequality is given.

9. A card is picked at random from a pack of 52 playing cards. Given that the picked card is a queen, the probability of this card to be a card of spade is

(*a*) $\frac{1}{3}$ (*b*) $\frac{4}{13}$ (*c*) $\frac{1}{4}$ (*d*) $\frac{1}{2}$

Sol. (*c*),

$$P(\text{spade/queen}) = \frac{P(\text{queen of spade})}{P(\text{queen})} = \frac{\frac{1}{52}}{\frac{4}{52}} = \frac{1}{4}$$

10. A die is thrown once. Let A be the event that the number obtained is greater than 3. Let B be the event that the number obtained is less than 5. Then $P(A \cup B)$ is

(*a*) $\frac{2}{5}$ (*b*) $\frac{3}{5}$ (*c*) 0 (*d*) 1

Sol. (*d*), as for event A : 4, 5, 6
and for event B : 1, 2, 3, 4

$$A \cap B : 4$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{6} + \frac{4}{6} - \frac{1}{6} = \frac{6}{6} = 1$$

Fill in the blanks in Questions from 11 to 15.

11. A relation in a set A is called _______ relation, if each element of A is related to itself.

Sol. reflexive

12. If $A + B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $A - 2B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, then A = ________.

Sol. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$, as

$$2(A + B) + (A - 2B) = 2\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$3A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2-1 & 0+1 \\ 2+0 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

13. The least value of the function $f(x) = ax + \frac{b}{x} (a > 0, b > 0, x > 0)$ is ________.

Sol. $2\sqrt{ab}$, as $\qquad f(x) = ax + \frac{b}{x}$

$$\Rightarrow \qquad f'(x) = a - \frac{b}{x^2}$$

For least value, $\qquad f'(x) = 0$

$$\Rightarrow \qquad a - \frac{b}{x^2} = 0 \Rightarrow x = \sqrt{\frac{b}{a}} \quad (x > 0)$$

$$f''(x) = \frac{2b}{x^3}$$

$$f''\left(\sqrt{\frac{b}{a}}\right) > 0$$

$$\therefore \qquad \text{Least value} = f\left(\sqrt{\frac{b}{a}}\right) = a \cdot \sqrt{\frac{b}{a}} + b \cdot \sqrt{\frac{a}{b}}$$

$$= \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$$

14. The integrating factor of the differential equation $x\frac{dy}{dx} + 2y = x^2$ is ________.

Or

The degree of the differential equation $1 + \left(\frac{dy}{dx}\right)^2 = x$ is ________.

Sol. x^2, as given $\qquad x\frac{dy}{dx} + 2y = x^2$

$$\Rightarrow \qquad \frac{dy}{dx} + \frac{2}{x}y = x$$

$$\text{Integrating factor} = e^{\int \frac{2}{x}dx} = e^{2\log x}$$

$$= e^{\log x^2} = x^2$$

Or

2, as given differential equation is $1 + \left(\frac{dy}{dx}\right)^2 = x$, highest order derivative is $\frac{dy}{dx}$. Can be written as polynomial of derivatives.

Degree = 2

15. The vector equation of a line which passes through the points (3, 4, –7) and (1, –1, 6) is ________.

Or

The line of shortest distance between two skew lines is ________ to both the lines.

Sol. $\vec{r} = (3\hat{i} + 4\hat{j} + -7\hat{k}) + \lambda(2\hat{i} + 5\hat{j} - 13\hat{k})$,

as DR's of line passing through the points (3, 4, –7) and (1, –1, 6) are 3 – 1, 4 + 1, –7 – 6 i.e. 2, 5, –13

$\therefore$ vector equation of line is, $\vec{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(2\hat{i} + 5\hat{j} - 13\hat{k})$

Or

Perpendicular

Question numbers 16 to 20 are of very short answer type questions.

16. Find the value of $\sin^{-1}\left[\sin\left(-\frac{17\pi}{8}\right)\right]$.

Sol.
$$\sin^{-1}\left[\sin\left(-\frac{17\pi}{8}\right)\right] = \sin^{-1}\left[\sin\left(-2\pi - \frac{\pi}{8}\right)\right]$$
$$= \sin^{-1}\left[-\sin\frac{\pi}{8}\right] = -\sin^{-1}\left[\sin\frac{\pi}{8}\right] = -\frac{\pi}{8}$$

17. For $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ write A^{-1}.

Sol. Given
$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$
$$|A| = \begin{vmatrix} 3 & -4 \\ 1 & -1 \end{vmatrix} = -3 + 4 = 1 \neq 0$$
$$\text{Adj}\, A = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$$
$\therefore$
$$A^{-1} = \frac{1}{|A|}\text{Adj}\, A = \frac{1}{1}\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$$

18. If the function f defined as $f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & , x \neq 3 \\ k & , x = 3 \end{cases}$ is continuous at $x = 3$, fmd the value of k.

Sol. For continuity at $x = 3$
$$\lim_{x \to 3} f(x) = f(3)$$
$\Rightarrow$
$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = k$$

$$\Rightarrow \quad \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)} = k$$

$$\Rightarrow \quad \lim_{x \to 3} (x+3) = k$$

$$\Rightarrow \quad 3 + 3 = k \Rightarrow k = 6$$

19. If $f(x) = x^4 - 10$, then find the approximate value of $f(2.1)$.

Or

Find the slope of the tangent to the curve $y = 2\sin^2(3x)$ at $x = \frac{\pi}{6}$.

Sol. $f(x) = x^4 - 10$

For approximation:

$$f(x + \Delta x) = f(x) + f'(x) \cdot \Delta x$$

$$= (x^4 - 10) + 4x^3 \cdot \Delta x$$

$$f(2 + 0.1) = (16 - 10) + 4 \times 8 \times 0.1$$

$$f(2.1) = 6 + 3.2 = 9.2$$

Or

Given $y = 2\sin^2(3x)$

$$\frac{dy}{dx} = 2 \times 2 \sin 3x \cos 3x \times 3 = 6 \sin 6x$$

$$\left.\frac{dy}{dx}\right]_{x = \frac{\pi}{6}} = 6 \sin \pi = 6 \times 0 = 0$$

Slope of the tangent at $x = \frac{\pi}{6}$ is 0.

20. Find the value of $\int_1^4 |x - 5| dx$.

Sol. Consider $\int_1^4 |x - 5| dx$

for $x < 5$, $|x - 5| = -(x - 5)$

$$\int_1^4 |x - 5|\, dx = \int_1^4 (5 - x) dx$$

$$= \left[5x - \frac{x^2}{2}\right]_1^4 = (20 - 8) - \left(5 - \frac{1}{2}\right)$$

$$= 12 - 5 + \frac{1}{2} = \frac{15}{2}$$

SECTION – B

Question numbers 21 to 26 carry 2 marks each.

21. If $f(x) = \dfrac{4x+3}{6x-4}, x \neq \dfrac{2}{3}$, then show that $(fof)(x) = x$, for all $x \neq \dfrac{2}{3}$. Also, write inverse of f.

Or

Check if the relation R in the set $I\!R$ of real numbers defined as

$R = \{(a, b) : a < b\}$ is (*i*) symmetric, (*ii*) transitive

Sol. Given $$f(x) = \frac{4x+3}{6x-4}$$

$$(fof)(x) = f[f(x)] = f\left(\frac{4x+3}{6x-4}\right)$$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} = \frac{16x+12+18x-12}{24x+18-24x+16}$$

$$= \frac{34x}{34} = x$$

As $$(fof)(x) = x \Rightarrow fof = I$$

$\Rightarrow$ $$f^{-1} = f$$

Or

Given relation, $$R = \{(a, b) : a < b\}$$

(*i*) For symmetric

Let $$a = 2, b = 3$$

$$(a, b) \in R \text{ as } 2 < 3$$

But $(b, a) \notin R$ as $3 \nless 2$

Hence, not symmetric as

$(a, b) \in R \not\Rightarrow (b, a) \in R$ for $a, b \in I\!R$

(*ii*) For transitive:

Let $$(a, b) \in R \text{ and } (b, c) \in R$$

$\Rightarrow$ $$a < b \text{ and } b < c \Rightarrow a < c$$

$\Rightarrow$ $$(a, c) \in R$$

Hence, transitive as

$$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$$

for all $a, b, c \in I\!R$

22. Find $\int \frac{x}{x^2+3x+2}dx$.

Sol. Let $$x = A(2x+3) + B = 2Ax + (3A+B)$$

Comparing the coefficients, we get

$$2A = 1,\ 3A + B = 0 \Rightarrow A = \frac{1}{2}, B = -\frac{3}{2}$$

$$\int \frac{x}{x^2+3x+2}dx = \int \frac{\frac{1}{2}(2x+3) - \frac{3}{2}}{x^2+3x+2}dx$$

$$= \frac{1}{2}\int \frac{2x+3}{x^2+3x+2}dx - \frac{3}{2}\int \frac{1}{x^2+3x+2}dx \qquad ...(i)$$

Consider $\frac{1}{2}\int \frac{2x+3}{x^2+3x+2}dx$ | Let $x^2 + 3x + 2 = t$ $\Rightarrow (2x+3)dx = dt$

$$= \frac{1}{2}\int \frac{1}{t}dt = \frac{1}{2}\log|t|$$

$$= \frac{1}{2}\log|x^2+3x+2| \qquad ...(ii)$$

Consider $$\frac{3}{2}\int \frac{1}{x^2+3x+2}dx = \frac{3}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \frac{9}{4} + 2}dx$$

$$= \frac{3}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}dx$$

$$= \frac{3}{2} \times \frac{1}{2\times\frac{1}{2}}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right|$$

$$= \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| \qquad ...(iii)$$

Substituting from (*ii*) and (*iii*) in (*i*), we get

$$\int \frac{x}{x^2+3x+2}dx = \frac{1}{2}\log|x^2+3x+2| - \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| + C$$

23. If $x = a\cos\theta$; $y = b\sin\theta$, then find $\frac{d^2y}{dx^2}$.

Or

Find the differential of $\sin^2 x$ w.r.t. $e^{\cos x}$.

Sol. Let $$x = a\cos\theta, y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta, \text{ and } \frac{dy}{d\theta} = b\cos\theta$$

$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$

$$\frac{d^2y}{dx^2} = -\frac{b}{a}(-\text{cosec}^2\,\theta)\cdot\frac{d\theta}{dx}$$

$$= \frac{b}{a}\text{cosec}^2\theta\cdot\frac{1}{-a\sin\theta} = -\frac{b}{a^2}\text{cosec}^3\,\theta$$

Or

Let $\quad y = \sin^2 x$ and $t = e^{\cos x}$

$\Rightarrow \quad \frac{dy}{dx} = 2\sin x\cos x$ and $\frac{dt}{dx} = -\sin x\, e^{\cos x}$

To find $\quad \frac{dy}{dt} = \frac{dy}{dx} \div \frac{dt}{dx}$

$\therefore \quad \frac{dy}{dt} = \frac{2\sin x\cos x}{-\sin x\cdot e^{\cos x}} = -2\cos x\cdot e^{-\cos x}$

24. Evaluate $\int_1^2 \left[\frac{1}{x} - \frac{1}{2x^2}\right] e^{2x} dx$.

Sol. Consider $\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x}\,dx = \frac{1}{2}\int_2^4 e^t\left(\frac{2}{t} - \frac{2}{t^2}\right)dt$

| Let $2x = t \Rightarrow 2dx = dt$, when $x = 1, t = 2$ and when $x = 2$, $t = 4$

$$= \frac{1}{2}\cdot 2\int_2^4 e^t\left(\frac{1}{t} - \frac{1}{t^2}\right)dt$$

$$= \left[\frac{1}{t}e^t\right]_2^4 = \left[\frac{1}{4}e^4 - \frac{1}{2}e^2\right]$$

$$= \frac{1}{4}e^4 - \frac{1}{2}e^2$$

25. Find the value of $\int_0^1 x(1-x)^n dx$.

Sol. Let $\quad I = \int_0^1 x(1-x)^n\,dx$

$\Rightarrow \quad I = \int_0^1 (1-x)\{1-(1-x)\}^n\,dx \quad \left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$

$$= \int_0^1 (1-x)(x)^n\,dx = \int_0^1 (x^n - x^{n+1})\,dx$$

$$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 = \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$= \frac{1}{(n+1)(n+2)}$$

26. Given two independent events A and B such that $P(A) = 0.3$ and $P(B) = 0.6$, find $P(A' \cap B')$.

Sol. A and B are independent events

$$P(A \cap B) = P(A) \cap P(B) = 0.3 \times 0.6 = 0.18$$
$$P(A' \cap B') = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 1 - (0.3 + 0.6 - 0.18) = 1 - 0.72 = 0.28$$

SECTION – C

Question numbers 27 to 32 carry 4 marks each.

27. Solve for x: $\sin^{-1}(1-x) - 2\sin^{-1}(x) = \frac{\pi}{2}$.

Sol. $\sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1}x$

$$\Rightarrow \quad 1 - x = \sin\left(\frac{\pi}{2} + 2\sin^{-1}x\right) = \cos(2\sin^{-1}x)$$
$$= 1 - 2\sin^2(\sin^{-1}x) \qquad [\text{using } \cos 2\theta = 1 - 2\sin^2\theta]$$
$$\Rightarrow \quad 1 - x = 1 - 2x^2$$
$$\Rightarrow \quad 2x^2 - x = 0 \Rightarrow x(2x-1) = 0$$
$$\Rightarrow \quad x = 0, \frac{1}{2} \Rightarrow x = 0, \text{ satisfies the given equation.}$$

Hence, $x = 0$.

28. If $y = (\log x)^x + x^{\log x}$, then find $\frac{dy}{dx}$.

Sol. Let $y = (\log x)^x + x^{\log x} = e^{x\log(\log x)} + e^{\log x \cdot \log x}$ $\qquad [x^a = e^{a\log x}]$

$$\frac{dy}{dx} = e^{x.\log(\log x)} \cdot \left\{\log(\log x)\cdot 1 + x\cdot\frac{1}{\log x}\cdot\frac{1}{x}\right\} + e^{\log x \cdot \log x} \cdot \left\{\frac{1}{x}\cdot\log x + \frac{1}{x}\cdot\log x\right\}$$
$$= (\log x)^x\left\{\log(\log x) + \frac{1}{\log x}\right\} + x^{\log x}\cdot\left(\frac{2}{x}\cdot\log x\right).$$

29. Solve the differential equation:

$$\boldsymbol{x\sin\left(\frac{y}{x}\right)\frac{dy}{dx} + x - y\sin\left(\frac{y}{x}\right) = 0}$$

Given that $x = 1$ when $y = \frac{\pi}{2}$.

Sol. Consider the equation,

$$x\cdot\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - x}{x\sin\left(\frac{y}{x}\right)} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) \qquad ...(i)$$

Consider

$$F(x, y) = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right)$$

$$F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \operatorname{cosec}\left(\frac{\lambda y}{\lambda x}\right)$$

$$= \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right)$$

$$= F(x, y)$$

Hence, function is homogeneous, so corresponding differential equation is homogeneous.

Let $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$$

From (i), we get

$$v + x\frac{dv}{dx} = v - \operatorname{cosec} v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\operatorname{cosec} v$$

$$\Rightarrow \quad \int \frac{dv}{\operatorname{cosec} v} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \int \sin v \, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad -\cos v = -\log|x| + C$$

$$\Rightarrow \quad \cos\left(\frac{y}{x}\right) = \log|x| - C \qquad ...(ii)$$

Given $x = 1$, when $y = \frac{\pi}{2}$

$$\Rightarrow \quad \cos\frac{\pi}{2} = \log|1| - C \Rightarrow C = 0$$

From (ii),

$$\cos\left(\frac{y}{x}\right) = \log|x| \text{ is the particular solution.}$$

30. If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

Or

Using vectors, find the area of the triangle *ABC* with vertices *A*(1, 2, 3), *B*(2, –1, 4) and *C*(4, 5, –1).

Sol. If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ represent two adjacent sides of a parallelogram then diagonals are along vectors $\vec{r}_1 = \vec{a} + \vec{b}$ and $\vec{r}_2 = \vec{a} - \vec{b}$.

$$\vec{r}_1 = \hat{i} + 2\hat{j} + 3\hat{k} + 2\hat{i} + 4\hat{j} - 5\hat{k}$$
$$= 3\hat{i} + 6\hat{j} - 2\hat{k}$$

unit vector along $\hat{r}_1 = \frac{\vec{r}_1}{|\vec{r}_1|} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{9+36+4}}$

$\Rightarrow$ $\hat{r}_1 = \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$

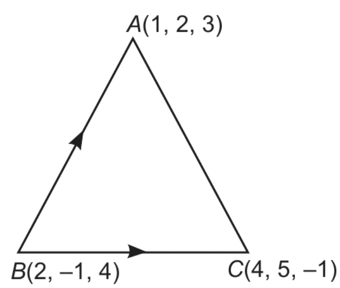
and $\vec{r}_2 = \hat{i} + 2\hat{j} + 3\hat{k} - 2\hat{i} - 4\hat{j} + 5\hat{k}$
$$= -\hat{i} - 2\hat{j} + 8\hat{k}$$

unit vector along $\hat{r}_2 = \frac{\vec{r}_2}{|\vec{r}_2|} = \frac{-\hat{i} - 2\hat{j} + 8\hat{k}}{\sqrt{1+4+64}}$

$\Rightarrow$ $\hat{r}_2 = \frac{-1}{\sqrt{69}}\hat{i} - \frac{2}{\sqrt{69}}\hat{j} + \frac{8}{\sqrt{69}}\hat{k}$

Or

Area of triangle $= \frac{1}{2}|\overrightarrow{BC} \times \overrightarrow{BA}|$



$$\overrightarrow{BC} = (4-2)\hat{i} + (5+1)\hat{j} + (-1-4)\hat{k} = 2\hat{i} + 6\hat{j} - 5\hat{k}$$
$$\overrightarrow{BA} = (1-2)\hat{i} + (2+1)\hat{j} + (3-4)\hat{k}$$
$$= -\hat{i} + 3\hat{j} - \hat{k}$$
$$\overrightarrow{BC} \times \overrightarrow{BA} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & -5 \\ -1 & 3 & -1 \end{vmatrix}$$
$$= \hat{i}(-6+15) - \hat{j}(-2-5) + \hat{k}(6+6)$$
$$= 9\hat{i} + 7\hat{j} + 12\hat{k}$$
$$|\overrightarrow{BC} \times \overrightarrow{BA}| = \sqrt{81+49+144} = \sqrt{274}$$

Area of triangle $= \frac{1}{2}\sqrt{274}$ sq units

31. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type *A* requires 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type *B* require 8 minutes each for cutting and 8 minutes each for assembling. Given that total time for cutting is 3 hours 20 minutes and for assembling 4 hours. The profit for type *A* souvenir is ₹ 100 each and for type *B* souvenir, profit is ₹ 120 each. How many souvenirs of each type should the company manufacture in order to maximize the profit? Formulate the problem as an LPP and solve its graphically.

Sol. Let x pieces of type A and y pieces of type B be produced.

	Cutting	**Assembling**	**Profit**
Type A	5 min	10 min	₹ 100
Type B	8 min	8 min	₹ 120
	$\leq$ 3 hrs 20 min	$\leq$ 4 hrs	

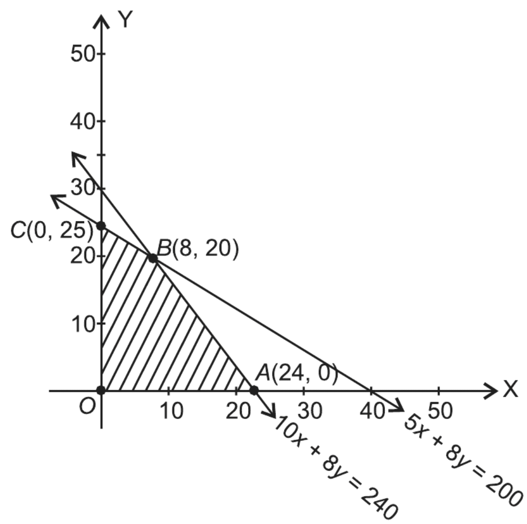
$\therefore$ *LPP* is to maximise profit $Z = 100x + 120y$

subject to constraints: $x \geq 0, y \geq 0$

$$5x + 8y \leq 200$$

$$10x + 8y \leq 240$$

On plotting the graph of inequations, we notice shaded portion represents the feasible solution.



Some points for maximum Z are $A(24, 0)$, $B(8, 20)$, $C(0, 25)$ and $O(0, 0)$.

Points	$Z = 100x + 120y$	**Values**	
$A(24, 0)$	2400 + 0	2400	
$B(8, 20)$	800 + 2400	3200	← Maximum
$C(0, 25)$	0 + 3000	3000	

We notice, profit is maximum at $B(8, 20)$, i.e. $x = 8, y = 20$. Hence, 8 souvenirs of type A and 20 souvenirs of type B be produced to get a maximum profit of ₹ 3200.

32. Three rotten apples are mixed with seven fresh apples. Find the probability distribution of the number of rotten apples, if three apples are drawn one by one with replacement. Find the mean of the number of rotten apples.

Or

In a shop X, 30 tins of ghee of type *A* and 40 tins of ghee of type *B* which look alike, are kept for sale. While in sbop Y, similar 50 tins of ghee of type *A* and 60 tins of ghee of type *B* are there. One tin of ghee is purchased from one of the randomly selected shop and is found to be of type *B*. Find the probability that it is purchased from shop *Y*.

Sol. Total apples: 3 rotten, 7 fresh.

X: Number of rotten apples

X can takes values 0, 1, 2, 3 as three apples are drawn one by one with replacement

S : getting a rotten apple

$$P(S) = \frac{3}{10}, P(\bar{S}) = \frac{7}{10}$$

$$P(0) = {}^3C_0[P(\bar{S})]^3 = 1 \times \left(\frac{7}{10}\right)^3 = \frac{343}{1000}$$

$$P(1) = {}^3C_1P(S)[P(\bar{S})]^2 = 3 \times \frac{3}{10} \times \frac{49}{100} = \frac{441}{1000}$$

$$P(2) = {}^3C_2[P(S)]^2\,P(\bar{S}) = 3 \times \left(\frac{3}{10}\right)^2 \cdot \frac{7}{10} = \frac{189}{1000}$$

$$P(3) = {}^3C_3[P(S)]^3 = 1\left(\frac{3}{10}\right)^3 = \frac{27}{1000}$$

Table for prbability distribution and mean

X	***P*(*X*)**	***X*·*P*(*X*)**
0	$\frac{343}{1000}$	0
1	$\frac{441}{1000}$	$\frac{441}{1000}$
2	$\frac{189}{1000}$	$\frac{378}{1000}$
3	$\frac{27}{1000}$	$\frac{81}{1000}$
	$\frac{1000}{1000} = 1$	$\frac{900}{1000} = \frac{9}{10}$

$$\text{Mean} = \Sigma X \cdot P(X) = \frac{9}{10}$$

Or

	Type A	**Type B**
Shop X:	30	40
Shop Y:	50	60

$$P(X) = \frac{1}{2}, P(Y) = \frac{1}{2}$$

E : tin purchased is found to be of type B

$$\therefore \quad P(E/X) = \frac{40}{70} = \frac{4}{7}, \quad P(E/Y) = \frac{60}{110} = \frac{6}{11}$$

Using Bayes' Theorem

Probability that tin of ghee of type B was purchased from shop Y is

$$P(Y/E) = \frac{P(Y)\cdot P(E/Y)}{P(X)\cdot P(E/X) + P(Y)\cdot P(E/Y)} = \frac{\frac{1}{2}\cdot\frac{6}{11}}{\frac{1}{2}\cdot\frac{4}{7}+\frac{1}{2}\cdot\frac{6}{11}} = \frac{\frac{6}{11}}{\frac{4}{7}+\frac{6}{11}}$$

$$= \frac{42}{44+42} = \frac{42}{86} = \frac{21}{43}$$

SECTION – D

Question 33 to 36 carry 6 marks each.

33. Find the vector and Cartesian equations of the line which is perpendicular to the lines with equations

$$\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4} \text{ and } \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and passes through the point (1, 1, 1). Also find the angle between the given lines.

Sol. Let equation of line through (1, 1, 1) be

$$\frac{x-1}{a} = \frac{y-1}{b} = \frac{z-1}{c} \quad ...(i)$$

Line (i) is perpendicular to the lines

$$\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4} \quad ...(ii)$$

and $$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad ...(iii)$$

$$\therefore \quad a + 2b + 4c = 0 \text{ and } 2a + 3b + 4c = 0$$

$$\Rightarrow \quad \frac{a}{8-12} = \frac{b}{8-4} = \frac{c}{3-4}$$

$\Rightarrow \qquad \dfrac{a}{-4} = \dfrac{b}{4} = \dfrac{c}{-1}$

DR's are –4, 4, –1 or 4, –4, 1

$\therefore$ Cartesian equation is

$$\frac{x-1}{4} = \frac{y-1}{-4} = \frac{z-1}{1} \qquad \text{[from } (i)]$$

and vector equation is

$$\vec{r} = (\hat{i} + \hat{j} + \hat{k}) + \lambda(4\hat{i} - 4\hat{j} + \hat{k})$$

Angle between lines (*ii*) and (*iii*) is

$$\cos\theta = \frac{1\times 2 + 2\times 3 + 4\times 4}{\sqrt{1+4+16}\,\sqrt{4+9+16}} = \frac{24}{\sqrt{21}\,\sqrt{29}}$$

$$\theta = \cos^{-1}\left(\frac{24}{\sqrt{609}}\right)$$

34. Using integration find the area of the region bounded between the two circles $x^2 + y^2 = 9$ and $(x-3)^2 + y^2 = 9$.

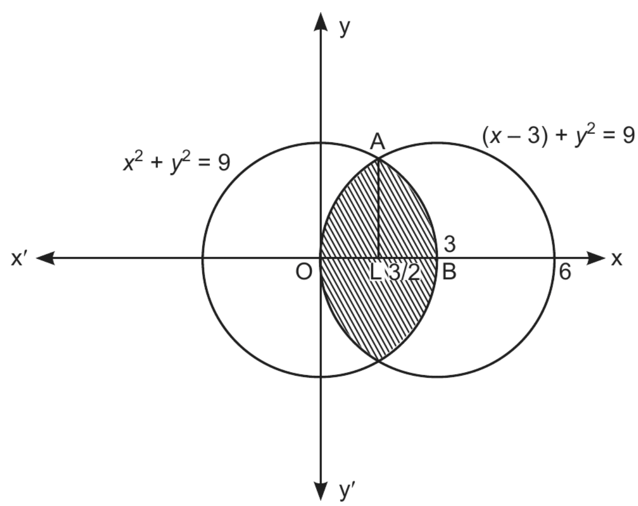
Or

Evaluate the following integral as the limit of sums $\int_1^4 (x^2 - x)dx$.

Sol. Given circles are $x^2 + y^2 = 9$ and $(x-3)^2 + y^2 = 9$

For first circle: centre (0, 0), radius = 3 and centre (3, 0), radius = 3, for second circle

On plotting the circles we notice we have to find the shaded area



Both the circles are symmetrical to the x-axis as both the functions are even with respect to y.

$\therefore \qquad \text{Area} = 2 \text{ area in 1st quadrant}$

$\qquad\qquad = 2[\text{ar}(OAL) + \text{ar}(LAB)]$

Eliminating y from two equations we get

$$9 - x^2 = 9 - (x-3)^2$$

$$\Rightarrow \quad -x^2 = -x^2 + 6x - 9 \Rightarrow x = \frac{3}{2}$$

$$\therefore \quad \text{Area} = 2\left[\int_0^{3/2} \sqrt{9-(x-3)^2}\,dx + \int_{3/2}^{3} \sqrt{9-x^2}\,dx\right]$$

$$= 2\left[\left\{\left(\frac{x-3}{2}\right)\sqrt{9-(x-3)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x-3}{3}\right)\right\}_0^{3/2} + \left\{\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right\}_{3/2}^{3}\right]$$

$$= 2\left[\left(\frac{-3}{4}\right)\sqrt{\frac{27}{4}} + \frac{9}{2}\sin^{-1}\left(-\frac{1}{2}\right)\right] - \left[0 + \frac{9}{2}\sin^{-1}(-1)\right] + \left[0 + \frac{9}{2}\sin^{-1}(1)\right] - \left[\frac{3}{4}\sqrt{\frac{27}{4}} + \frac{9}{2}\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= 2\left[-\left(\frac{3}{4}\right)\times\frac{3\sqrt{3}}{2} - \frac{9}{2}\cdot\frac{\pi}{6} + \frac{9}{2}\cdot\frac{\pi}{2} + \frac{9}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{3\sqrt{3}}{2} - \frac{9}{2}\cdot\frac{\pi}{6}\right]$$

$$= \left(6\pi - \frac{9\sqrt{3}}{2}\right) \text{ sq units}$$

Or

Here, $a = 1, b = 4, f(x) = x^2 - x, h = \dfrac{4-1}{n} \Rightarrow nh = 3$ ...(*i*)

$$\int_1^4 (x^2 - x)dx = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + ... + f(1+(n-1)h)]$$

$$f(1) = 1 - 1 = 0$$

$$f(1+h) = (1+h)^2 - (1+h) = 1 + h^2 + 2h - 1 - h = h^2 + h$$

$$f(1+2h) = (1+2h)^2 - (1+2h) = 1 + 4h^2 + 4h - 1 - 2h = 4h^2 + 2h$$

$$f\{1+(n-1)h\} = \{1+(n-1)h\}^2 - \{1+(n-1)h\}$$

$$= 1 + (n-1)^2h^2 + 2(n-1)h - 1 - (n-1)h$$

$$= (n-1)^2h^2 + (n-1)h$$

$$\int_1^4 (x^2 - x)dx = \lim_{h\to 0} h[(0) + (h^2 + h) + (4h^2 + 2h) + ... + \{(n-1)^2h^2 + (n-1)h\}]$$

$$= \lim_{h\to 0} h[h^2\{1 + 4 + ... + (n-1)^2\} + h\{1 + 2 + ... + (n-1)\}]$$

$$= \lim_{h\to 0} h\left[h^2 . \frac{(n-1)n(2n-1)}{6} + h . \frac{(n-1)n}{2}\right]$$

$$= \lim_{h\to 0}\left[\frac{(nh-h)(nh)(2nh-h)}{6} + \frac{(nh-h)(nh)}{2}\right]$$

$$= \lim_{h\to 0}\left[\frac{(3-h)(3)(6-h)}{6} + \frac{(3-h)3}{2}\right] = \frac{3\times3\times6}{6} + \frac{9}{2} = 9 + \frac{9}{2} = \frac{27}{2} \quad \text{[From } (i)\text{]}$$

35. Find the minimum value of $(ax + by)$, where $xy = c^2$.

Sol. Given $$xy = c^2 \quad ...(i)$$

Let $$Z = ax + by$$

$$\Rightarrow \quad Z = ax + \frac{bc^2}{x} \quad ...(ii)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dZ}{dx} = a - \frac{bc^2}{x^2}$$

For minimum Z, $\frac{dZ}{dx} = 0 \Rightarrow a - \frac{bc^2}{x^2} = 0$

$$\Rightarrow \quad x^2 = \frac{bc^2}{a} \Rightarrow x = \sqrt{\frac{bc^2}{a}}$$

$$\frac{d^2Z}{dx^2} = 0 + \frac{2bc^2}{x^3} = \frac{2bc^2}{x^3}$$

$$\left.\frac{d^2Z}{dx^2}\right]_{x = \sqrt{\frac{bc^2}{a}}} = \frac{2bc^2}{\left(\frac{bc^2}{a}\right)^{3/2}} > 0$$

Hence, for $x = \sqrt{\frac{bc^2}{a}}$, Z is minimum

Substituting in (ii), we get

Minimum $$Z = a \cdot \sqrt{\frac{bc^2}{a}} + \frac{bc^2}{\sqrt{\frac{bc^2}{a}}}$$

$$= \sqrt{abc^2} + \sqrt{abc^2} = 2 \times \sqrt{abc^2}$$

36. If a, b, c are p^{th}, q^{th} and r^{th} terms respectively of a G.P., then prove that

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$$

Or

If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1}.

Using A^{-1}, solve the following system of equations:

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

Sol. Let A be first term, R the common ratio of a given G.P.

Also, $a_p = a, a_q = b, a_r = c$

$\Rightarrow \quad a = AR^{p-1} \Rightarrow \log a = \log A + (p-1)\log R$

$\Rightarrow \quad b = AR^{q-1} \Rightarrow \log b = \log A + (q-1)\log R$

$\Rightarrow \quad c = AR^{r-1} \Rightarrow \log c = \log A + (r-1)\log R$

Consider $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$

$$= \begin{vmatrix} \log A & p & 1 \\ \log A & q & 1 \\ \log A & r & 1 \end{vmatrix} + \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix} \quad \text{...}(i)$$

$$= \log A \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} + \log R \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}$$

$$= \log A \times 0 + \log R \begin{vmatrix} 0 & p & 1 \\ 0 & q & 1 \\ 0 & r & 1 \end{vmatrix} \quad [\text{on performing } C_1 \to C_1 - C_2 + C_3]$$

$$= 0 + \log R \times 0$$

$$= 0$$

Or

Consider $\quad A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2(0) + 3(-2) + 5(1) = -1 \neq 0$$

Hence A^{-1} exists.

Matrix formed by cofactors of each element in $|A|$

$$\begin{bmatrix} A_{11} = (-4+4) = 0, & A_{12} = -(-6+4) = 2, & A_{13} = (3-2) = 1 \\ A_{21} = -(6-5) = -1, & A_{22} = (-4-5) = -9, & A_{23} = -(2+3) = -5 \\ A_{31} = (12-10) = 2, & A_{32} = -(-8-15) = 23, & A_{33} = (4+9) = 13 \end{bmatrix}$$

$$\text{Adj}\, A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj} A = -\frac{Q}{Q}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \quad ...(i)$$

Consider equations,

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Corresponding matrix equation is

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

i.e. $AX = B$, Its solution is $X = A^{-1}B$

$$\Rightarrow \quad X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

or
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad x = 1, y = 2$ and $z = 3$ is solution.

SET-II [UNCOMMON QUESTIONS TO SET-I]

1. If $[x \;\; 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$, then x equals

(*a*) 0 **(*b*) – 2** **(*c*) –1** **(*d*) 2**

Sol. (*d*), Consider $[x \quad 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$

$$\Rightarrow \quad [x - 2 \qquad 0 + 0] = [0 \qquad 0]$$

$$\Rightarrow \quad x - 2 = 0 \Rightarrow x = 2$$

2. $\int 4^x 3^x dx$ equals

(*a*) $\frac{12^x}{\log 12} + C$ **(*b*) $\frac{4^x}{\log 4} + C$** **(*c*) $\left(\frac{4^x \cdot 3^x}{\log 4 \cdot \log 3}\right) + C$** **(*d*) $\frac{3^x}{\log 3} + C$**

Sol. (*a*), Consider $\int 4^x 3^x dx = \int (12)^x dx$

$$= \frac{(12)^x}{\log_e 12} + C$$

3. A number is chosen randomly from numbers 1 to 60. The probability that the chosen number is a multiple of 2 or 5 is

(*a*) $\frac{2}{5}$ (*b*) $\frac{3}{5}$ (*c*) $\frac{7}{10}$ (*d*) $\frac{9}{10}$

Sol. (*b*), A : multiple of 2 B : Multiple of 5

$A \cap B$: Multiple of 10

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{30}{60} + \frac{12}{60} - \frac{6}{60} = \frac{36}{60} = \frac{3}{5}$$

11. A relation R on a set A is called _______, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for $a_1, a_2, a_3 \in A$.

Sol. Transitive

16. Evaluate: $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$.

Sol. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right]$

$$= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right] = \sin\frac{\pi}{2} = 1.$$

17. Using differential, find the approximate value of $\sqrt{36.6}$ upto 2 decimal places.

Or

Find the slope of tangent to the curve $y = 2\cos^2(3x)$ at $x = \frac{\pi}{6}$.

Sol. Let $f(x) = \sqrt{x}$, $f(x + \Delta x) = \sqrt{x + \Delta x}$, Δx is small increment in x and $f'(x) = \frac{1}{2\sqrt{x}}$

Using approximation

$$f(x + \Delta x) = f(x) + f'(x) \cdot \Delta x$$

$$\Rightarrow \quad \sqrt{x + \Delta x} = \sqrt{x} + \frac{1}{2\sqrt{x}} \cdot \Delta x$$

Let $x = 36$, $\Delta x = 0.6$

$$\therefore \quad \sqrt{36.6} = \sqrt{36} + \frac{1}{2\sqrt{36}} \times 0.6$$

$$= 6 + \frac{1}{12} \times 0.6 = 6 + 0.05 = 6.05$$

Or

Consider $$y = 2\cos^2(3x)$$

$$\frac{dy}{dx} = 2 \cdot 2\cos(3x) \cdot \{-3\sin 3x\}$$

$$= -6\sin 6x$$

$$\left.\frac{dy}{dx}\right]_{x=\frac{\pi}{6}} = -6\sin\pi = -6 \times 0 = 0$$

$\therefore$ Slope of the tangent at $x = \dfrac{\pi}{6}$ is 0.

21. Find $\int \frac{x+1}{x(1-2x)}dx$.

Sol. Let $$\frac{x+1}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x}$$

$\Rightarrow$ $$x + 1 = A(1-2x) + Bx$$

$$= x(-2A + B) + A$$

Comparing the coefficients, we get

$$-2A + B = 1, A = 1 \Rightarrow B = 3$$

$\therefore$ $$\int \frac{x+1}{x(1-2x)}dx = \int\left(\frac{1}{x} + \frac{3}{1-2x}\right)dx$$

$$= \log|x| - \frac{3}{2}\log|1-2x| + C$$

22. Evaluate $\int \frac{x\sin^{-1}(x^2)}{\sqrt{1-x^4}}dx$.

Sol. Consider $$\int \frac{x\sin^{-1}x^2}{\sqrt{1-x^4}}dx = \frac{1}{2}\int t\,dt$$

| Let $\sin^{-1}x^2 = t$
| $\Rightarrow \dfrac{2x}{\sqrt{1-x^4}}dx = dt$

$$= \frac{1}{4}t^2 + C$$

$$= \frac{1}{4}(\sin^{-1}x^2)^2 + C$$

27. Prove that $\tan\left[2\tan^{-1}\left(\frac{1}{2}\right) - \cot^{-1}3\right] = \frac{9}{13}$.

Sol. $$\tan\left[2\tan^{-1}\frac{1}{2} - \cot^{-1}3\right] = \tan\left[\tan^{-1}\frac{2\times\frac{1}{2}}{1-\frac{1}{4}} - \tan^{-1}\frac{1}{3}\right]$$

$$= \tan\left[\tan^{-1}\frac{4}{3} - \tan^{-1}\frac{1}{3}\right]$$

$$= \tan\left[\tan^{-1}\frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{3}\cdot\frac{1}{3}}\right]$$

$$= \tan\left[\tan^{-1}\frac{9}{13}\right] = \frac{9}{13}$$

28. If $y = (\cos x)^x + \tan^{-1}\sqrt{x}$, find $\frac{dy}{dx}$.

Sol. Consider $y = (\cos x)^x + \tan^{-1}\sqrt{x}$

$\Rightarrow$ $y = e^{x\log(\cos x)} + \tan^{-1}\sqrt{x}$ $\quad [e^{a\log x} = x^a]$

$$\Rightarrow \quad \frac{dy}{dx} = e^{x\log(\cos x)}\cdot\left\{x\cdot\frac{1}{\cos x}(-\sin x)+\log(\cos x)\cdot 1\right\} + \frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}$$

$$= (\cos x)^x\{-x\tan x + \log(\cos x)\} + \frac{1}{2\sqrt{x}(1+x)}$$

36. Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, 1).

Sol. Let point (x, y) on the curve $y^2 = 4x$ is nearest to the point (2, 1)

$$\text{Distance, } D = \sqrt{(x-2)^2+(y-1)^2}$$

If D is minimum then D^2 is also minimum.

$$\therefore \quad D^2 = E \text{ (say)}$$

$$= (x-2)^2 + (y-1)^2$$

As point (x, y) lies on the curve $y^2 = 4x$

$$\Rightarrow \quad x = \frac{y^2}{4} \quad ...(i)$$

$$\therefore \quad E = \left(\frac{y^2}{4}-2\right)^2 + (y-1)^2$$

$$= \frac{y^4}{16} - y^2 + 4 + y^2 - 2y + 1$$

$$E = \frac{y^4}{16} - 2y + 5$$

$$\frac{dE}{dy} = \frac{4y^3}{16} - 2$$

$$= \frac{y^3}{4} - 2$$

For minimum distance $\frac{dE}{dy} = 0$

$\Rightarrow$ $\frac{y^3}{4} - 2 = 0$

$\Rightarrow$ $y^3 = 8$

$= (2)^3$

$\Rightarrow$ $y = 2$

$$\frac{d^2E}{dy^2} = \frac{3y^2}{4}$$

$$\left.\frac{d^2E}{dy^2}\right]_{y=2} = \frac{3 \times 4}{4} > 0$$

$\therefore$ for $y = 2$, distance is minimum

Substituting in (i), we get $x = 1$

$\therefore$ point (1, 2) on the curve is nearest to the point (2, 1).

SET-III [UNCOMMON QUESTIONS TO SET-I AND SET-II]

1. It A is a skew symmetric matrix of order 3, then the value of $|A|$ is

(a) 3 **(b) 0** **(c) 9** **(d) 27**

Sol. (b), let

$$|A| = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

$$= 0 - a(0 + bc) + b(ac - 0) = 0$$

6. If $y = \log_e\left(\frac{x^2}{e^2}\right)$, then $\frac{d^2y}{dx^2}$ equals

(a) $-\frac{1}{x}$ **(b) $-\frac{1}{x^2}$** **(c) $\frac{2}{x^2}$** **(d) $-\frac{2}{x^2}$**

Sol. (d), Consider $y = \log_e\left(\frac{x^2}{e^2}\right)$

$= \log_e x^2 - \log_e e^2$

$\Rightarrow$ $y = 2\log_e x - 2$ [$\because \log_e e = 1$]

$\therefore$ $\frac{dy}{dx} = 2 \cdot \frac{1}{x} - 0 = \frac{2}{x}$

and $\frac{d^2y}{dx^2} = -\frac{2}{x^2}$

9. The distance of the origin (0, 0, 0) from the plane $-2x + 6y - 3z = -7$ is

(*a*) 1 unit **(*b*) $\sqrt{2}$ units** **(*c*) $2\sqrt{2}$ units** **(*d*) 3 units**

Sol. (*a*), Distance $= \left|\dfrac{(-2)\times 0 + 6\times 0 - 3\times 0 + 7}{\sqrt{4+36+9}}\right| = 1$ unit

11. If A and B are square matrices each of order 3 and $|A| = 5$, $|B| = 3$, then the value of $|3AB|$ is ________.

Sol. 405, as

$$\begin{aligned}|3AB| &= 3^3|AB| \\ &= 27|A||B| \\ &= 27 \times 5 \times 3 \\ &= 405\end{aligned}$$

16. Find the cofactors of all the elements of $\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$.

Sol. Determinant corresponding to the given matrix is $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$.

Cofactor of a_{ij}

$A_{11} = (-1)^{1+1} \cdot 3 = 3,$ $A_{12} = (-1)^{1+2}(4) = -4$

$A_{21} = (-1)^{2+1}(-2) = 2,$ $A_{22} = (-1)^{2+2}(1) = 1$

17. Let $f(x) = x|x|$, for all $x \in R$ check its differentiability at $x = 0$.

Sol.

$$f(x) = x|x| = \begin{cases} x^2 & , x \geq 0 \\ -x^2 & , x < 0 \end{cases}$$

$$\underset{x=0}{\text{LHD}} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-h^2 - 0}{-h}$$

$$= \lim_{h \to 0} h = 0$$

$$\underset{x=0}{\text{RHD}} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 - 0}{h} = \lim_{h \to 0} h = 0$$

As $\underset{x=0}{\text{LHD}} = \underset{x=0}{\text{RHD}}$

Hence, functions is differentiable at $x = 0$

21. Find $\int \frac{x+1}{(x+2)(x+3)}dx$.

Sol. Let $$\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$\Rightarrow$ $$x + 1 = A(x + 3) + B(x + 2)$$
$$= x(A + B) + (3A + 2B)$$

Comparing coefficients, we get

$$A + B = 1, 3A + 2B = 1$$

$\Rightarrow$ $$A = -1, B = 2$$

$\therefore$ $$\int \frac{x+1}{(x+2)(x+3)}dx = -\int \frac{1}{x+2}dx + \int \frac{2}{x+3}dx$$
$$= -\log|x + 2| + 2\log|x + 3| + C$$

26. Find the value of $\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx$.

Sol. $$\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx = \int_0^1 \tan^{-1}\left\{\frac{(1-x)-x}{1+x(1-x)}\right\}dx$$

$$= \int_0^1 \tan^{-1}(1-x)dx - \int_0^1 \tan^{-1}x\,dx \quad ...(i)$$

$$\int_0^1 \tan^{-1}(1-x)dx = \int_0^1 \tan^{-1}\{1-(1-x)\}dx$$

[Using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$]

$$= \int_0^1 \tan^{-1}x\,dx$$

From (*i*), we get

$$\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right)dx = \int_0^1 \tan^{-1}x\,dx - \int_0^1 \tan^{-1}x\,dx = 0$$

27. Solve the equation $x : \sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$ $(x \neq 0)$.

Sol. Consider $$\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}, (x \neq 0) \quad ...(i)$$

$\Rightarrow$ $$\sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{5}{x}\right) = \cos^{-1}\left(\frac{5}{x}\right)$$

$\Rightarrow$ $$\sin^{-1}\left(\frac{12}{x}\right) = \sin^{-1}\sqrt{1-\frac{25}{x^2}}$$

$$\Rightarrow \quad \frac{12}{x} = \sqrt{1-\frac{25}{x^2}} \Rightarrow \frac{144}{x^2} = 1 - \frac{25}{x^2}$$

$$\Rightarrow \quad \frac{169}{x^2} = 1 \Rightarrow x^2 = 169 \Rightarrow x = \pm 13$$

From (i), $x = 13$

28. Find the general solution of the differential equation

$ye^{x/y}dx = (xe^{x/y} + y^2)dy, y \neq 0$

Sol. Consider equation

$$ye^{x/y}dx = (xe^{x/y} + y^2)dy, y \neq 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{xe^{x/y} + y^2}{ye^{x/y}}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x}{y} + \frac{y}{e^{x/y}} \quad \quad ...(i)$$

Let $\quad x = vy \Rightarrow \frac{dx}{dy} = v + y \cdot \frac{dv}{dy}$

From (i), we get

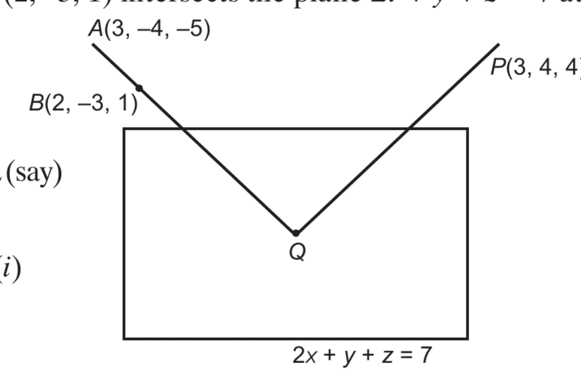
$$v + y \cdot \frac{dv}{dy} = v + \frac{y}{e^v}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{y}{e^v} \Rightarrow \int e^v dv = \int dy$$

$$\Rightarrow \quad e^v = y + C \Rightarrow e^{x/y} = y + C$$

33. Find the distance of the point $P(3, 4, 4)$ from the point, where the line joining the points $A(3, -4, -5)$ and $B(2, -3, 1)$ intersects the plane $2x + y + z = 7$.

Sol. Let line through the points $A(3, -4, -5)$ and $B(2, -3, 1)$ intersects the plane $2x + y + z = 7$ at point Q.



Equation of AB is $\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}$

i.e. $\quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda \text{ (say)}$

General point on the line is

$Q(-\lambda + 3, \lambda - 4, 6\lambda - 5) \quad \quad ...(i)$

If this point lies on the plane $2x + y + z = 7$,

then $2(-\lambda + 3) + \lambda - 4 + 6\lambda - 5 = 7$

$\Rightarrow -2\lambda + 6 + \lambda - 4 + 6\lambda - 5 = 7 \Rightarrow 5\lambda = 10 \Rightarrow \lambda = 2$

Substituting in (i), the point of intersection is $Q(1, -2, 7)$

$\therefore$ Distance $PQ = \sqrt{(3-1)^2 + (4+2)^2 + (4-7)^2} = \sqrt{4 + 36 + 9} = 7$ units

Examination Papers, 2019

[Dehli Set-I, II, III]

Time allowed: 3 Hours] [Maximum Marks: 100

General Instructions:

(i) *All questions are compulsory.*

(ii) *This question paper contains **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of 8 questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.*

(iii) *All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.*

(iv) *There is no overall choice. However, internal choice has been provided in 1 question of Section A, **3** questions of Section B, **3** questions of Section C and **3** questions of Section D. You have to attempt only **one** of the alternatives in all such questions.*

(v) *Use of calculators is not permitted. You may ask logarithmic tables, if required.*

Note: * Marked Questions are out of course for the forthcoming examinations.

SET-I

SECTION – A

Question numbers* 1 *to* 4 *carry* 1 *mark each.

1. If A and B are square matrices of the same order 3, such that $|A| = 2$ and $AB = 2I$, write the value of $|B|$.

Sol. $AB = 2I \Rightarrow |AB| = |2I| \Rightarrow |A||B| = 2^3|I| \Rightarrow 2 \cdot |B| = 8 \times 1 \Rightarrow |B| = 4.$

2. If $f(x) = x + 1$, find $\frac{d}{dx}(fof)(x)$.

Sol. $(fof)(x) = f\{f(x)\} = f(x+1) = x + 1 + 1 = x + 2$

$$\frac{d}{dx}(fof)(x) = \frac{d}{dx}(x+2) = 1.$$

3. Find the order and the degree of the differential equation

$$x^2\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4.$$

Sol. Consider $x^2 \cdot \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4$

Highest order derivative is $\frac{d^2y}{dx^2}$

$\therefore$ Order = 2, degree = 1.

4. If a line makes angles 90°, 135°, 45° with the *x*, *y* and *z*-axes respectively, find its direction cosines.

Or

Find the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the vector $2\hat{i} + 2\hat{j} - 3\hat{k}$.

Sol. Given $\alpha = 90°, \beta = 135°, \gamma = 45°$

$\therefore$ Direction cosines are $\cos\alpha, \cos\beta, \cos\gamma$, i.e. $\cos 90°, \cos 135°, \cos 45°$, i.e. $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

Or

Line passes through the point (3, 4, 5) i.e. point with position vector $3\hat{i} + 4\hat{j} + 5\hat{k}$ and parallel to vector $2\hat{i} + 2\hat{j} - 3\hat{k}$. Equation is $\vec{r} = (3\hat{i} + 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 2\hat{j} - 3\hat{k})$.

SECTION – B

Question numbers 5 to 12 carry 2 marks each.

***5. Examine whether the operation * defined on *R* by $a * b = ab + 1$ is (*i*) a binary or not. (*ii*) if a binary operation, is it associative or not?**

6. Find a matrix *A* such that $2A - 3B + 5C = O$, where $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$.

Sol. $2A - 3B + 5C = O \Rightarrow 2A = 3B - 5C = 3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$

$\Rightarrow 2A = \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} - \begin{bmatrix} 10 & 0 & -10 \\ 35 & 5 & 30 \end{bmatrix} = \begin{bmatrix} -6- & 10 & 6-0 & 0+10 \\ 9- & 35 & 3-5 & 12-30 \end{bmatrix}$

$\Rightarrow 2A = \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix} \Rightarrow A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$.

7. Find : $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx$

Sol. Consider $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx = \int \frac{1}{\sqrt{t^2 + 4}} dt$ |Let $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$= \log|t + \sqrt{t^2 + 4}| + C = \log|\tan x + \sqrt{\tan^2 x + 4}| + C.$

8. Find : $\int \sqrt{1-\sin 2x}\, dx, \frac{\pi}{4} < x < \frac{\pi}{2}$

Or

Find : $\int \sin^{-1}(2x)\, dx$

Sol. $\int \sqrt{1-\sin 2x}\; dx = \int \sqrt{\cos^2 x + \sin^2 x - 2\sin x \cos x}\; dx = \int \sqrt{(\sin x - \cos x)^2}\; dx;$

$= \int (\sin x - \cos x)\, dx$ for $\frac{\pi}{4} < x < \frac{\pi}{2}$ $(\sin x > \cos x)$

$= -\cos x - \sin x + C$

Or

Consider $\int \sin^{-1}(2x)\, dx = \frac{1}{2}\int \sin^{-1} t\; dt = \frac{1}{2}\left[\sin^{-1} t \,.\, 1 - \int \frac{1}{\sqrt{1-t^2}} \,.\, t\; dt\right]$ | Let $2x = t$ $\Rightarrow 2dx = dt$

$= \frac{1}{2}\left[\sin^{-1} t - \int \frac{t}{\sqrt{1-t^2}}\; dt\right]$ ...(*i*)

Consider $\int \frac{t}{\sqrt{1-t^2}}\, dt$ | Let $1 - t^2 = y$

$= -\frac{1}{2}\int \frac{1}{\sqrt{y}}\, dy$ | $\Rightarrow \; -2t\; dt = dy$

$= -\frac{1}{2} \,.\, 2\sqrt{y} = -\sqrt{y} = -\sqrt{1-t^2}$ | $\Rightarrow \; t\; dt = -\frac{1}{2} dy$

From (*i*), $\int \sin^{-1}(2x)\, dx = \frac{1}{2}[\sin^{-1} t + \sqrt{1-t^2}] = \frac{1}{2}[\sin^{-1}(2x) + \sqrt{1-4x^2}] + C.$

9. Form the differential equation representing the family of curves $y = e^{2x}(a + bx)$, where 'a' and 'b' are arbitrary constants.

Sol. Consider $y = e^{2x}(a + bx)$ ...(*i*)

Differentiating with respect to x, we get

$$\frac{dy}{dx} = e^{2x}(b) + 2.e^{2x}(a + bx)$$

$$\Rightarrow \quad \frac{dy}{dx} = b\, e^{2x} + 2y \qquad \text{[from } (i)] \ldots (ii)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 2b\, e^{2x} + 2\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\left[\frac{dy}{dx} - 2y\right] + 2\,\frac{dy}{dx} \qquad \text{[from } (ii)]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0 \text{ is the required equation.}$$

10. If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

Or

If $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$, find $[\vec{a} \ \ \vec{b} \ \ \vec{c}]$.

Sol. Let $\hat{a}$ and $\hat{b}$ be unit vectors, given

$$|\hat{a}| = 1, |\hat{b}| = 1 \text{ and } |\hat{a} + \hat{b}| = 1$$

Now, $|\hat{a} + \hat{b}|^2 = 1 \Rightarrow (\hat{a} + \hat{b})^2 = 1$

$\Rightarrow$ $\hat{a}^2 + \hat{b}^2 + 2\hat{a}\cdot\hat{b} = 1$

$\Rightarrow$ $2\hat{a}\cdot\hat{b} = 1 - 2 = -1$ ...(*i*)

Consider, $|\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b})^2$

$= \hat{a}^2 + \hat{b}^2 - 2\hat{a}\cdot\hat{b} = 1 + 1 - (-1)$ [from (*i*)]

$\Rightarrow$ $|\hat{a} - \hat{b}|^2 = 3 \Rightarrow |\hat{a} - \hat{b}| = \sqrt{3}$.

Or

Given $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$

$$\therefore \ [\vec{a} \ \ \vec{b} \ \ \vec{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 2(-4-1) - 3(2+3) + 1(1-6) = -10 - 15 - 5 = -30.$$

11. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let *A* be the event "number is even" and *B* be the event "number is marked red". Find whether the events *A* and *B* are independent or not.

Sol. Die : 1, 2, 3 marked in red; 4, 5, 6 marked in green

A : number is even, i.e. 2, 4, 6; $P(A) = \frac{3}{6} = \frac{1}{2}$

B : number is marked red, i.e. 1, 2, 3; $P(B) = \frac{3}{6} = \frac{1}{2}$

$A \cap B$: number is even and red, i.e. 2; $P(A \cap B) = \frac{1}{6}$

As $P(A \cap B) \neq P(A)\,P(B)$, i.e. $\frac{1}{6} \neq \frac{1}{2}\cdot\frac{1}{2}$.

Hence, not independent.

12. The random variable X has a probability distribution $P(X)$ of the following form, where '*k*' is some number.

$$P(X = x) = \begin{cases} k & , \text{ if } x = 0 \\ 2k & , \text{ if } x = 1 \\ 3k & , \text{ if } x = 2 \\ 0 & , \text{ otherwise} \end{cases}$$

Determine the value of '*k*'.

Sol. For the probability distribution,

$\Sigma P(X = x) \Rightarrow P(0) + P(1) + P(2) + P(3) + ... = 1$

$\Rightarrow\ k + 2k + 3k + 0 + ... = 1\ \Rightarrow\ k = \dfrac{1}{6}$

SECTION – C

Question numbers* 13 *to* 23 *carry* 4 *marks each.

13. **Show that the relation R on $\mathbb{R}$, the set of real numbers defined as $R = \{(a, b) : a \le b\}$, is reflexive, and transitive but not symmetric.**

Or

Prove that the function $f : N \to N$, defined by $f(x) = x^2 + x + 1$ is one-one but not onto. Find inverse of $f : N \to S$, where S is range of f.

Sol. Given relation $R = \{(a, b) : a \le b\}$

For reflexive: Let for $a \in \mathbb{R}$

$(a, a) \in R\ \Rightarrow\ a \le a$, true. Hence, reflexive.

For symmetric: Let $a, b \in \mathbb{R}$

such that $(a, b) \in R\ \Rightarrow\ a \le b$

This may not imply $b \le a$

e.g. $(2, 3) \in R$ i.e. $2 \le 3$ but $(3, 2) \notin R$

Hence, not symmetric.

For transitive: Let $a, b, c \in \mathbb{R}$

such that $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow a \le b$ and $b \le c \Rightarrow a \le c$ (transitive law)

$\Rightarrow (a, c) \in R$

Hence, transitive.

Or

Consider function $f : N \to N$ given by $f(x) = x^2 + x + 1$

For one-one: Let for $x_1, x_2 \in N$

$$f(x_1) = f(x_2) \Rightarrow x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$$

$\Rightarrow\quad (x_1^2 - x_2^2) + (x_1 - x_2) = 0 \quad\Rightarrow\quad (x_1 - x_2)(x_1 + x_2 + 1) = 0$

$\Rightarrow\quad x_1 - x_2 = 0$ as $x_1 + x_2 + 1 \ne 0,\ x_1, x_2 \in N$

$\Rightarrow\quad x_1 = x_2$

$\therefore\quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Hence, one-one.

For onto: Let for $y \in N$ (co-domain), there exists $x \in N$ (domain) such that $y = f(x)$

$\Rightarrow\quad y = x^2 + x + 1$

$\Rightarrow\quad x^2 + x + (1 - y) = 0$

$$x = \frac{-1 \pm \sqrt{1-4(1)(1-y)}}{2} = \frac{-1 \pm \sqrt{1-4+4y}}{2} = \frac{-1 \pm \sqrt{4y-3}}{2}$$

may not belong to N.

e.g. if $y = 1$, then $x = \frac{-1 \pm \sqrt{1}}{2} = \frac{-1 \pm 1}{2} \notin N$.

Hence not onto

For inverse: $f : N \to S$, S is range of f i.e. each element of S is associated with some element N.

Now $\quad x = \frac{-1 \pm \sqrt{4y-3}}{2} \Rightarrow x = \frac{\sqrt{4y-3}-1}{2}$, as $x = \frac{-1-\sqrt{4y-3}}{2} \notin N$

$$\therefore \quad f^{-1}(y) = \frac{\sqrt{4y-3}-1}{2}$$

$$\therefore \quad f^{-1}(x) = \frac{\sqrt{4x-3}-1}{2}.$$

14. Solve: $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$.

Sol. Consider $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$ ...(*i*)

$\Rightarrow \quad \tan^{-1}\left(\frac{4x+6x}{1-24x^2}\right) = \frac{\pi}{4}$ $\quad [24x^2 < 1]$

$\Rightarrow \quad \frac{10x}{1-24x^2} = \tan\frac{\pi}{4} = 1$

$\Rightarrow 10x = 1 - 24x^2 \quad \Rightarrow \quad 24x^2 + 10x - 1 = 0 \quad \Rightarrow \quad 24x^2 + 12x - 2x - 1 = 0$

$\Rightarrow 12x(2x+1) - 1(2x+1) = 0$

$\Rightarrow (12x-1)(2x+1) = 0$

$\Rightarrow 12x - 1 = 0$ or $2x + 1 = 0$

$\Rightarrow x = \frac{1}{12}$ or $x = -\frac{1}{2}$

$x = \frac{1}{12}$ satisfy but $x = -\frac{1}{2}$, does not satisfy (*i*), Hence $x = \frac{1}{12}$.

15. Using properties of determinants, prove that $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$.

Sol. Let $\Delta = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$ [by performing $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$]

$= (a-1)^2 \begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$ [by taking $(a-1)$ common from R_1 and R_2]

$= (a-1)^2 [0 - 0 + 1(a+1-2)] = (a-1)^3$ [by expanding along C_3]

16. **If $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.**

Or

If $x^y - y^x = a^b$, find $\frac{dy}{dx}$.

Sol. Given, $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{x^2+y^2}\cdot\frac{d}{dx}(x^2+y^2) = 2\cdot\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{2x+2y\frac{dy}{dx}}{x^2+y^2} = 2\cdot\frac{1}{1+\frac{y^2}{x^2}}\cdot\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)$$

$$\Rightarrow \quad \frac{x+y\frac{dy}{dx}}{x^2+y^2} = \frac{1}{x^2+y^2}\left(x\frac{dy}{dx}-y\right)$$

$$\Rightarrow \quad x+y\frac{dy}{dx} = x\frac{dy}{dx}-y$$

$$\Rightarrow \quad (x-y)\frac{dy}{dx} = x+y \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x+y}{x-y}.$$

Or

Consider $\qquad x^y - y^x = a^b$

$$\Rightarrow \quad e^{\log x^y} - e^{\log y^x} = a^b \qquad [a^b \text{ is constant}]$$

$$\Rightarrow \quad e^{y\log x} - e^{x\log y} = a^b$$

Differentiating with respect to x, we get

$$\Rightarrow \quad e^{y\log x}\cdot\left\{y\cdot\frac{1}{x}+\log x\cdot\frac{dy}{dx}\right\} - e^{x\log y}\left\{x\cdot\frac{1}{y}\cdot\frac{dy}{dx}+\log y\cdot 1\right\} = 0$$

$$\Rightarrow \quad x^y\left\{\frac{y}{x}+\log x\cdot\frac{dy}{dx}\right\} - y^x\left\{\frac{x}{y}\cdot\frac{dy}{dx}+\log y\right\} = 0$$

$$\Rightarrow \quad y\cdot x^{y-1} + x^y\cdot\log x\cdot\frac{dy}{dx} - x.y^{x-1}\cdot\frac{dy}{dx} - y^x.\log y = 0$$

$$\Rightarrow \quad \frac{dy}{dx}\{x^y\log x - xy^{x-1}\} = y^x\log y - y\cdot x^{y-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^x\log y - y\,x^{y-1}}{x^y\log x - x\,y^{x-1}}$$

17. If $y = (\sin^{-1} x)^2$, prove that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2 = 0$.

Sol. Consider $y = (\sin^{-1} x)^2$

$$\Rightarrow \frac{dy}{dx} = 2\sin^{-1}x\frac{1}{\sqrt{1-x^2}} = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} \quad \Rightarrow \quad \sqrt{1-x^2}\,\frac{dy}{dx} = 2\sin^{-1}x.$$

Again differentiating both sides w.r.t. x, we get

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\,\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow \qquad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0.$$

18. Find the equation of tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$. Also, write the equation of normal to the curve at the point of contact.

Sol. Given $y = \sqrt{3x-2}$ ...(i)

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{3x-2}}\cdot 3 = \frac{3}{2\sqrt{3x-2}} \quad \text{[Slope of tangent at } (x, y)]$$

Tangent is parallel to the line $4x - 2y + 5 = 0$, slope of the line = 2

$$\Rightarrow \qquad \frac{3}{2\sqrt{3x-2}} = 2 \text{ (slopes are equal)}$$

$$\Rightarrow \qquad 9 = 16(3x-2) \Rightarrow x = \frac{41}{48}$$

Substituting in curve (i), we get

$$y = \sqrt{\frac{41-32}{16}} = \frac{3}{4}, \ \therefore \text{ point is } \left(\frac{41}{48}, \frac{3}{4}\right)$$

$\therefore$ equation of the tangent parallel to the line $4x - 2y + 5 = 0$ is

$$y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right),$$

i.e. $\qquad 48x - 24y - 23 = 0$

Also, $\qquad$ Slope of normal $= -\dfrac{1}{\text{slope of tangent}} = -\dfrac{1}{2}$

$\therefore$ Equation of normal is, $\quad y - \dfrac{3}{4} = -\dfrac{1}{2}\left(x - \dfrac{41}{48}\right)$

$$\Rightarrow \qquad \frac{4y-3}{4} = -\frac{1}{2}\left(\frac{48x-41}{48}\right) \quad \Rightarrow \quad 96y - 72 = -48x + 41$$

$$\Rightarrow \qquad 48x + 96y - 113 = 0.$$

19. Find: $\int \frac{3x+5}{x^2+3x-18}\,dx$

Sol. Consider $\int \frac{3x+5}{x^2+3x-18}\,dx$

Let $3x + 5 = A \cdot \frac{d}{dx}(x^2 + 3x - 18) + B = A(2x + 3) + B$

$\Rightarrow \quad 3x + 5 = x(2A) + (3A + B)$

Comparing the coefficients of x and constant term, we get

$$2A = 3 \quad \text{and} \quad 3A + B = 5$$

$$\Rightarrow \quad A = \frac{3}{2}, \quad B = \frac{1}{2}$$

$$\therefore \quad \int \frac{3x+5}{x^2+3x-18}\,dx = \frac{3}{2}\int \frac{2x+3}{x^2+3x-18}\,dx + \frac{1}{2}\int \frac{1}{x^2+3x-18}\,dx \quad ...(i)$$

Consider $\int \frac{2x+3}{x^2+3x-18}\,dx$

| Let $x^2 + 3x - 18 = t$
| $\Rightarrow (2x + 3)\,dx = dt$

$$= \int \frac{1}{t}\,dt = \log|t| + C_1 = \log|x^2 + 3x - 18| + C_1 \quad ...(ii)$$

Consider $\int \frac{1}{x^2+3x-18}\,dx = \int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\,dx$

$$= \frac{1}{2\times\frac{9}{2}} \log\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right|$$

| $x^2 + 3x - 18$
| $= \left(x+\frac{3}{2}\right)^2 - \frac{9}{4} - 18$
| $= \left(x+\frac{3}{2}\right)^2 - \frac{81}{4}$

$$= \frac{1}{9}\log\left|\frac{x-3}{x+6}\right| + C_2 \quad ...(iii)$$

Substituting from (*ii*) and (*iii*) in (*i*), we get

$$\int \frac{3x+5}{x^2+3x-18}\,dx = \frac{3}{2}\log|x^2+3x-18| + \frac{1}{18}\log\left|\frac{x-3}{x+6}\right| + C. \text{ Where } C_1 + C_2 = \text{C(constant)}$$

20. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$**, hence evaluate** $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$.

Sol. Consider, $\int_0^a f(a-x)dx = -\int_a^0 f(t)\,dt$

| Let $a - x = t \Rightarrow -dx = dt$
| When $x = 0$, $t = a$ and when $x = a$, $t = 0$

$$= \int_0^a f(t)dt \sim \int_0^a f(x)\,dx$$

Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ ...(*i*)

$I = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1+\cos^2 (\pi - x)} dx$ $\left[\text{using property } \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$ ...(*ii*)

$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$ [by adding (*i*) and (*ii*)]

$\Rightarrow \quad 2I = -\pi \int_1^{-1} \frac{1}{1+t^2} dt = -\pi \left[\tan^{-1} t\right]_1^{-1}$

Let $\cos x = t$
$\Rightarrow -\sin x\, dx = dt$
when $x = 0, t = 1$
and when $x = \pi, t = -1$

$= -\pi [\tan^{-1} (-1) - \tan^{-1} (1)]$

$2I = -\pi\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{2} \quad \Rightarrow \quad I = \frac{\pi^2}{4}$

21. Solve the differential equation: $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$, given that $y = 0$ when $x = 1$.

Or

Solve the differential equation : $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$, subject to the initial condition $y(0) = 0$.

Sol. Consider equation $xdy - ydx = \sqrt{x^2+y^2}\,dx$

$\Rightarrow \quad xdy = (y + \sqrt{x^2 + y^2})dx \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x}$ ...(*i*)

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x} = v + \sqrt{1+v^2}$ [from (*i*)]

$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2}$

$\Rightarrow \quad \int \frac{1}{\sqrt{1+v^2}} dv = \int \frac{1}{x} dx$

$\Rightarrow \quad \log | v + \sqrt{1+v^2} | = \log |x| + \log C$

$\Rightarrow \quad \log | v + \sqrt{1+v^2} | = \log Cx \Rightarrow \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = Cx$

$\Rightarrow \quad y + \sqrt{x^2+y^2} = Cx^2$ ...(*i*)

Given $\quad y = 0$ when $x = 1$

$\Rightarrow \qquad 0 + \sqrt{1+0} = C\,.\,(1)^2 \quad \Rightarrow \quad C = 1$

$\therefore$ From (i), $\quad y + \sqrt{x^2+y^2} = x^2$ is the solution.

Or

Consider equation $\qquad (1 + x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$

$$\Rightarrow \qquad \frac{dy}{dx} + \frac{2x}{1+x^2}\,y = \frac{4x^2}{1+x^2}$$

Here $\quad P(x) = \dfrac{2x}{1+x^2}$, $Q(x) = \dfrac{4x^2}{1+x^2}$

Integrating factor $= e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = (1 + x^2)$

$\therefore$ Solution in $\qquad$ (I.F.)$y = \int \{I.F.\}Q(x)\}dx$

$$(1 + x^2)y = \int (1 + x^2)\,.\,\frac{4x^2}{1+x^2}dx = 4\int x^2 dx$$

$$(1 + x^2)y = \frac{4x^3}{3} + C \qquad ...(i)$$

Given $\qquad y(0) = 0$, i.e. when $x = 0$, $\quad y = 0$

$$(1 + 0)0 = \frac{4}{3} \times 0 + C \Rightarrow C = 0$$

$\therefore$ from (i), $(1 + x^2)y = \dfrac{4x^3}{3}$ is the required solution.

22. If $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or not.

Sol. Given $A(\hat{i} + \hat{j} + \hat{k})$, $B(2\hat{i} + 5\hat{j})$, $C(3\hat{i} + 2\hat{j} - 3\hat{k})$ and $D(\hat{i} - 6\hat{j} - \hat{k})$

$$\overrightarrow{AB} = (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 4\hat{j} - \hat{k}$$

$$\overrightarrow{CD} = (\hat{i} - 6\hat{j} - \hat{k}) - (3\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{i} - 8\hat{j} + 2\hat{k}$$

Let θ be angle between straight lines AB and CD then

$$\cos\theta = \frac{(\hat{i} + 4\hat{j} - \hat{k})\cdot(-2\hat{i} - 8\hat{j} + 2\hat{k})}{\sqrt{1+16+1}\sqrt{4+64+4}} = \frac{-2-32-2}{\sqrt{18}\cdot\sqrt{72}}$$

$$= \frac{-36}{36} = -1$$

$\Rightarrow \qquad \theta = \pi \quad \Rightarrow \quad \overrightarrow{AB} = \lambda\overrightarrow{CD}$,

Here $\lambda = -2 \quad \Rightarrow \quad \overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or parallel vectors.

23. Find the value of λ, so that the lines $\frac{1-x}{3} = \frac{7y-14}{\lambda} = \frac{z-3}{2}$ and $\frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not.

Sol. Given, L_1: $\frac{1-x}{3} = \frac{7y-14}{\lambda} = \frac{z-3}{2}$, i.e. $\frac{x-1}{-3} = \frac{7(y-2)}{\lambda} = \frac{z-3}{2}$

$$\Rightarrow \quad \frac{x-1}{-21} = \frac{y-2}{\lambda} = \frac{z-3}{14}$$

L_2: $\frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$, i.e. $\frac{-7(x-1)}{3\lambda} = \frac{y-5}{1} = \frac{z-6}{-5}$

$$\Rightarrow \quad \frac{x-1}{3\lambda} = \frac{y-5}{-7} = \frac{z-6}{35}$$

If lines are perpendicular, then

$(-21)(3\lambda) + (\lambda)(-7) + (14)(35) = 0 \quad \Rightarrow \quad 9\lambda + \lambda - 70 = 0 \quad \Rightarrow \quad 10\lambda = 70 \quad \Rightarrow \quad \lambda = 7$

$\therefore$ lines are L_1: $\frac{x-1}{-21} = \frac{y-2}{7} = \frac{z-3}{14}$ or $\frac{x-1}{-3} = \frac{y-2}{1} = \frac{z-3}{2}$

L_2: $\frac{x-1}{21} = \frac{y-5}{-7} = \frac{z-6}{35}$ or $\frac{x-1}{3} = \frac{y-5}{-1} = \frac{z-6}{5}$

General point on L_1 is $(-3\lambda + 1, \lambda + 2, 2\lambda + 3)$ ...(*i*)

General point on L_2 is $(3\mu + 1, -\mu + 5, 5\mu + 6)$ ...(*ii*)

If lines interest then for some λ, μ, (*i*) and (*ii*) represent the same point.

i.e. $-3\lambda + 1 = 3\mu + 1 \quad \Rightarrow \quad -\lambda = \mu$ ...(*iii*)

$\lambda + 2 = -\mu + 5 \quad \Rightarrow \quad \lambda + \mu = 3$ ...(*iv*)

$2\lambda + 3 = 5\mu + 6 \quad \Rightarrow \quad 2\lambda - 5\mu = 3$ ...(*v*)

From (*iii*) and (*iv*), we get $\lambda + \mu = 0$ and $\lambda + \mu = 3$, not possible

$\therefore$ No value of λ, μ will make (*iii*), (*iv*) and (*v*) true.

Hence, lines do not intersect.

SECTION – D

Question numbers 24 to 29 carry 6 marks each.

24. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, find A^{-1}. Hence, solve the system of equations $x + y + z = 6, x + 2z = 7, 3x + y + z = 12$.

Or

Find the inverse of the following matrix using elementary operations.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Sol. Consider $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{Adj} A \qquad \text{....}(i)$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 1(-2) -1\,(-5) + 1(1) = -2 + 5 + 1 = 4 \neq 0$$

Hence, A^{-1} exists.

Matrix formed by cofactors of elements in $|A|$

$A_{11} = (0 - 2) = -2, \quad A_{12} = -(1 - 6) = 5, \quad A_{13} = (1 - 0) = 1$

$A_{21} = -(1 - 1) = 0, \quad A_{22} = (1 - 3) = -2, \quad A_{23} = -(1 - 3) = 2$

$A_{31} = (2 - 0) = 2, \quad A_{32} = -(2 - 1) = -1, \quad A_{33} = (0 - 1) = -1$

$$\text{Adj} A = \begin{bmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{bmatrix}' = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\therefore \text{ from } (i), A^{-1} = \frac{1}{4} \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \qquad \text{...}(ii)$$

Consider equations

$$x + y + z = 6$$
$$x + 0y + 2z = 7$$
$$3x + y + z = 12$$

Corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

i.e. $AX = B$

Its solution is $X = A^{-1}B$

$$\Rightarrow \quad X = \frac{1}{4} \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix} \qquad [\text{from } (ii)]$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -12 + 0 + 24 \\ 30 - 14 - 12 \\ 6 + 14 - 12 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$\Rightarrow x = 3, y = 1, z = 2$ is solution.

Or

$$\text{Consider } A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Let $A = IA$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad \text{[by performing } R_2 \to R_2 + R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A \quad \text{[by performing } R_2 \to R_2 + 2R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -4 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A \quad \text{[by performing } R_1 \to R_1 - 2R_2 \text{ and } R_3 \to R_3 + 2R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A \quad \text{[by performing } R_1 \to R_1 + 2R_3]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}.$$

25. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square metre for the sides, what is the cost of least expensive tank?

Sol. Let dimension of tank be x m, y m and 2m

Volume of the tank $V = 2xy = 8 \Rightarrow xy = 4$ ...(i)

$$\text{Cost, } C = 70\,xy + 2(x + y) \times 2 \times 45$$

$$= 70 \times 4 + 180\left(x + \frac{4}{x}\right) = 280 + 180\left(x + \frac{4}{x}\right) \quad \text{[using } (i)] \ldots(ii)$$

$$\frac{dC}{dx} = 180\left(1 - \frac{4}{x^2}\right)$$

$$\text{For minimum cost } \frac{dC}{dx} = 0 \Rightarrow 1 - \frac{4}{x^2} = 0 \Rightarrow x = 2 \ (x \neq -2)$$

$$\frac{d^2C}{dx^2} = 180 \times \frac{8}{x^3} \Rightarrow \left.\frac{d^2C}{dx^2}\right]_{x=2} > 0$$

$\therefore$ cost C is minimum for $x = 2$

$y = 2$ [from (i)]

From (ii), least cost $= 280 + 180\left(2 + \frac{4}{2}\right) = 280 + 720 =$ ₹ 1000.

26. Using integration, find the area of triangle ABC, whose vertices are $A(2, 5)$, $B(4, 7)$ and $C(6, 2)$.

Or

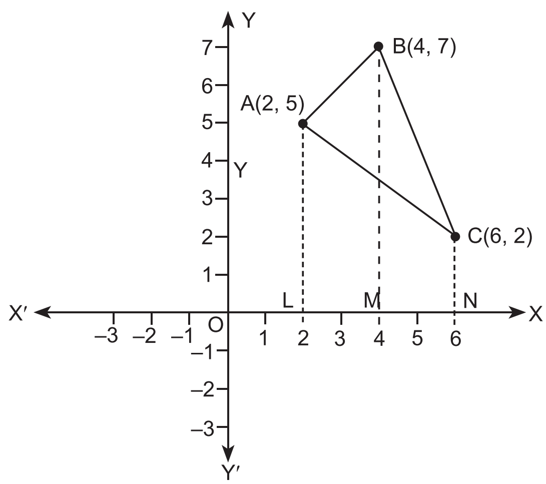
Find the area of the region lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

Sol. Given vertices $A(2, 5)$, $B(4, 7)$ and $C(6, 2)$

$\text{ar}(ABC) = \text{ar}(LABM) + \text{ar}(MBCN) - \text{ar}(LACN)$ ...(*i*)

For $\text{ar}(LABM)$: $A(2, 5)$, $B(4, 7)$

Equation of AB: $y - 5 = \frac{7-5}{4-2}(x-2)$



$\Rightarrow \quad y - 5 = x - 2 \quad \Rightarrow \quad y = x + 3$

$$\Rightarrow \quad \text{ar}(LABM) = \int_2^4 (x+3)\,dx = \left[\frac{x^2}{2} + 3x\right]_2^4 = \left(\frac{16}{2} + 12\right) - \left(\frac{4}{2} + 6\right)$$

$= 20 - 8 = 12$ ...(*ii*)

For $\text{ar}(MBCN)$: $B(4, 7)$, $C(6, 2)$

Equation of BC : $y - 7 = \frac{2-7}{6-4}(x-4)$

$\Rightarrow \quad y = -\frac{5}{2}x + 10 + 7 \quad \Rightarrow \quad y = -\frac{5}{2}x + 17$

$$\therefore \quad \text{ar}(MBCN) = \int_4^6 \left(-\frac{5}{2}x + 17\right)dx = \left[-\frac{5x^2}{4} + 17x\right]_4^6$$

$$= \left(-\frac{5\times 36}{4} + 102\right) - \left(-\frac{5\times 16}{4} + 68\right)$$

$= (-45 + 102) - (-20 + 68) = 57 - 48 = 9$ ...(*iii*)

For $\text{ar}(LACN)$: $A(2, 5)$, $C(6, 2)$

Equation of AC : $y - 5 = \frac{2-5}{6-2}(x-2)$

$\Rightarrow \quad y - 5 = -\frac{3}{4}(x-2) \quad \Rightarrow \quad y - 5 = -\frac{3}{4}x + \frac{3}{2}$

$\Rightarrow \quad y = -\frac{3}{4}x + \frac{3}{2} + 5$

$\Rightarrow \quad y = -\frac{3}{4}x + \frac{13}{2}$

$$\therefore \quad \text{ar}(LACN) = \int_2^6 \left(-\frac{3}{4}x + \frac{13}{2}\right) dx = \left[-\frac{3x^2}{8} + \frac{13}{2}x\right]_2^6$$

$$= \left(\frac{-3\times 36}{8} + \frac{13}{2}\times 6\right) - \left(\frac{-3\times 4}{8} + \frac{13}{2}\times 2\right)$$

$$= \left(\frac{-27}{2} + 39\right) - \left(-\frac{3}{2} + 13\right) = -\frac{27}{2} + 39 + \frac{3}{2} - 13$$

$$= -12 + 39 - 13 = 14 \quad ...(iv)$$

Substituting from (*ii*), (*iii*) and (*iv*) in (*i*), we get

$$\text{ar}(ABC) = 12 + 9 - 14 = 7 \text{ sq units.}$$

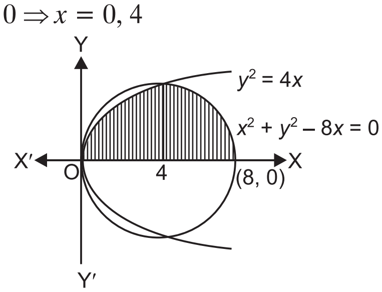
Or

Circle is $x^2 + y^2 - 8x = 0$, i.e. $(x-4)^2 + (y-0)^2 = (4)^2$ and parabola $y^2 = 4x$

Centre of the circle is (4, 0), radius is 4.

For point of intersection of the circle and parabola

$x^2 + 4x = 8x \Rightarrow x^2 - 4x = 0 \Rightarrow x = 0, 4$



$$\text{Area} = \int_0^4 \sqrt{4x}\, dx + \int_4^8 \sqrt{16-(x-4)^2}\, dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^4 + \left[\frac{x-4}{2}\sqrt{16-(x-4)^2} + 8\sin^{-1}\frac{x-4}{4}\right]_4^8$$

$$= \left[\frac{4}{3}(4)^{\frac{3}{2}} - 0\right] + [0 + 8\sin^{-1}1] - [0+0]$$

$$= \left(\frac{32}{3} + 8\cdot\frac{\pi}{2}\right)$$

$$= \frac{32}{3} + 8\cdot\frac{\pi}{2} = \left(4\pi + \frac{32}{3}\right) \text{ sq units.}$$

27. Find the vector and Cartesian equations of the plane passing through the points (2, 2, –1), (3, 4, 2) and (7, 0, 6). Also, find the vector equation of a plane passing through (4, 3, 1) and parallel to the plane obtained above.

Or

Find the vector equation of the plane that contains the line $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and the point (–1, 3, –4). Also, find the length of the perpendicular drawn from the point (2, 1, 4) to the plane, thus obtained.

Sol. Cartesian equation of the plane passing through (2, 2, –1), (3, 4, 2) and (7, 0, 6) is

$$\begin{vmatrix} x-2 & y-2 & z+1 \\ 3-2 & 4-2 & 2+1 \\ 7-2 & 0-2 & 6+1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} = 0$$

$\Rightarrow (x-2)(20) - (y-2)(-8) + (z+1)(-12) = 0$

$\Rightarrow \quad 20x - 40 + 8y - 16 - 12z - 12 = 0$

$\Rightarrow \quad 20x + 8y - 12z - 68 = 0 \Rightarrow 5x + 2y - 3z - 17 = 0$

Vector equation of the plane is $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) - 17 = 0$ ...(*i*)

Vector equation of the plane parallel to plane (*i*) is

$\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda = 0$ ...(*ii*)

If plane (*ii*) passes through point (4, 3, 1), i.e. point with position vector $(4\hat{i} + 3\hat{j} + \hat{k})$, then

$(4\hat{i} + 3\hat{j} + \hat{k}) \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda = 0$

$\Rightarrow \quad 20 + 6 - 3 + \lambda = 0 \Rightarrow \lambda = -23$

$\therefore$ Vector equation of plane is $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) - 23 = 0$

Or

Vector equation of the plane containing line $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ is

$[\vec{r} - (\hat{i} + \hat{j})] \cdot \vec{n} = 0$ ...(*i*)

and $(\hat{i} + 2\hat{j} - \hat{k}) \cdot \vec{n} = 0$ ...(*ii*)

Plane (*i*) contains the point (–1, 3, –4), i.e. point with position vector $(-\hat{i} + 3\hat{j} - 4\hat{k})$

$\therefore \quad [-\hat{i} + 3\hat{j} - 4\hat{k} - \hat{i} - \hat{j}] \cdot \vec{n} = 0$

$\Rightarrow \quad (-2\hat{i} + 2\hat{j} - 4\hat{k}) \cdot \vec{n} = 0$ ...(*iii*)

From (*ii*) and (*iii*), we get

$\vec{n} = (\hat{i} + 2\hat{j} - \hat{k}) \times (-2\hat{i} + 2\hat{j} - 4\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 2 & -4 \end{vmatrix} = \hat{i}(-6) - \hat{j}(-6) + \hat{k}(6) = -6\hat{i} + 6\hat{j} + 6\hat{k}$$

$\therefore$ Equation of plane is $[\vec{r} - (\hat{i} + \hat{j})] \cdot (-6\hat{i} + 6\hat{j} + 6\hat{k}) = 0$ [from (*i*)]

i.e. $\vec{r} \cdot (-6\hat{i} + 6\hat{j} + 6\hat{k}) - (-6 + 6 + 0) = 0$

$\Rightarrow$ $\vec{r} \cdot (-\hat{i} + \hat{j} + \hat{k}) = 0$ is required equation of the plane.

Length of perpendicular drawn from point (2, 1, 4), i.e. point with position vector $(2\hat{i} + \hat{j} + 4\hat{k})$ is

$$\left| \frac{(2\hat{i} + \hat{j} + 4\hat{k}) \cdot (-\hat{i} + \hat{j} + \hat{k})}{\sqrt{1+1+1}} \right| = \left| \frac{-2+1+4}{\sqrt{3}} \right| = \sqrt{3} \text{ units.}$$

28. A manufacturer has three machine operators *A*, *B* and *C*. The first operator *A* produces 1% of defective items, whereas the other two operators *B* and *C* produces 5% and 7% defective items respectively. *A* is on the job for 50% of the time, *B* on the job 30% of the time and *C* on the job for 20% of the time. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by *A*?

Sol. $P(A) = \frac{50}{100} = \frac{1}{2}, P(B) = \frac{30}{100} = \frac{3}{10}, P(C) = \frac{20}{100} = \frac{1}{5}$

E : item chosen is defective

$P(E/A) = \frac{1}{100}, P(E/B) = \frac{5}{100}, P(E/C) = \frac{7}{100}$

Using Bayes' theorem, probability of defective item produced by *A*,

$$P(A/E) = \frac{P(A) \,.\, P(E/A)}{P(A) \,.\, P(E/A) + P(B) \,.\, P(E/B) + P(C) \,.\, P(E/C)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{100}}{\frac{1}{2} \cdot \frac{1}{100} + \frac{3}{10} \cdot \frac{5}{100} + \frac{1}{5} \cdot \frac{7}{100}}$$

$$= \frac{5}{5 + 15 + 14} = \frac{5}{34}$$

29. A manufacturer has employed 5 skilled men and 10 semi-skilled men and makes two models *A* and *B* of an article. The making of one item of model *A* requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. One item of model *B* requires 1 hour by a skilled man and 3 hours by a semi-skilled man. No man is expected to work more than 8 hours per day. The manufacturer's profit on an item of model *A* is ₹ 15 and on an item of model *B* is ₹ 10. How many of items of each model should be made per day in order to maximise daily profit? Formulate the above *LPP* and solve it graphically and find the maximum profit.

Sol.

	Skilled men	**Semi-skilled-men**	**Profit (in ₹)**
Model *A*	2 hr	2 hr	15
Model *B*	1 hr	3 hr	10
	$\leq 5 \times 8$, i.e. 40 hr	$\leq 10 \times 8$, i.e. 80 hr	

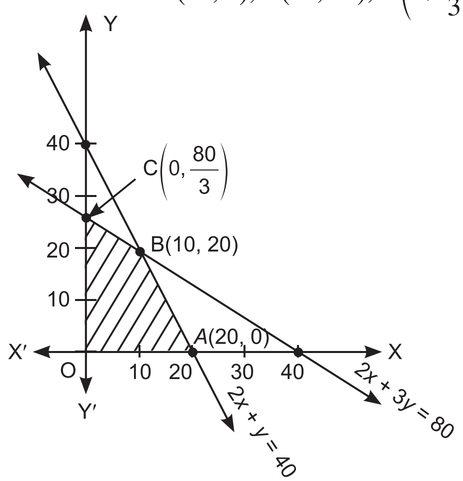
Let x items of model A and y items of model B are made. Then LPP is

To maximise profit $Z = 15x + 10y$

Subject to the constraints

$$x \geq 0, y \geq 0$$
$$2x + y \leq 40$$
$$2x + 3y \leq 80$$

Plotting the graph of inequations we notice we have shaded portion as feasiable solution. Possible points for maximum Z are $A(20, 0)$, $B(10, 20)$, $C\left(0, \frac{80}{3}\right)$.



Points	$Z = 15x + 10y$	Values	
$A(20, 0)$	$300 + 0$	300	
$B(10, 20)$	$150 + 200$	350	← Maximum
$C\left(0, \frac{80}{3}\right)$	$0 + \frac{800}{3}$	266.7	

Z is maximum for $B(10, 20)$, i.e. $x = 10, y = 20$

Hence, 10 items of type A and 20 of type B must be made for a maximum profit of ₹ 350.

SET-II (Uncommon questions to Set-I)

SECTION – A

2. If $f(x) = x + 7$ and $g(x) = x - 7, x \in R$, then find $\frac{d}{dx}(fog)(x)$.

Sol. Given $f(x) = x + 7$ and $g(x) = x - 7$

$$(fog)(x) = f\{g(x)\} = f(x - 7) = x - 7 + 7 = x$$

$$\therefore \quad \frac{d}{dx}[(fog)(x)] = \frac{d}{dx}(x) = 1$$

3. Find the value of $x - y$, if

$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Sol. Consider $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$\Rightarrow \quad 2 + y = 5, 2x + 2 = 8 \Rightarrow y = 3$ and $x = 3$

$x - y = 3 - 3 = 0.$

SECTION – B

6. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find $(A^2 - 5A)$.

Sol.

$$A^2 - 5A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1-0 & 1-3+0 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10 & -1-0 & 2-5 \\ 9-10 & -2-5 & 5-15 \\ 0-5 & -1+5 & -2-0 \end{bmatrix} = \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix}$$

12. Find: $\int \frac{\tan^2 x \sec^2 x}{1 - \tan^6 x} dx$.

Sol. Consider $\int \frac{\tan^2 x \sec^2 x}{1 - \tan^6 x} dx$

Let $\tan^3 x = t$
$\Rightarrow 3\tan^2 x \sec^2 x dx = dt$

$= \frac{1}{3}\int \frac{1}{1 - t^2} dt = \frac{1}{3} \times \frac{1}{2} \log\left|\frac{1+t}{1-t}\right| + C$

$= \frac{1}{6}\log\left|\frac{1 + \tan^3 x}{1 - \tan^3 x}\right| + C$

SECTION – C

13. Solve for x : $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

Sol. Consider $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$ ...(*i*)

$\Rightarrow \quad \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right) = \frac{\pi}{4}$ $[6x^2 < 1]$

$\Rightarrow \quad \frac{5x}{1-6x^2} = \tan\left(\frac{\pi}{4}\right) = 1$

$\Rightarrow \quad 1 - 6x^2 = 5x \quad \Rightarrow \quad 6x^2 + 5x - 1 = 0$

$\Rightarrow \quad x = \frac{-5 \pm \sqrt{25+24}}{12}$

$\Rightarrow \quad x = \frac{-5 \pm 7}{12} \Rightarrow x = -1 \quad \text{or} \quad \frac{1}{6}$

$x = \frac{1}{6}$ satisfies equation (*i*) and $x = -1$ does not satisfy (*i*).

$\therefore \quad x = \frac{1}{6}$ is solution.

18. Using properties of determinants, prove the following:

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$$

Sol. Consider $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$

$= \begin{vmatrix} a+b & -c-b & -b \\ a+b & b+c & -a \\ -b-a & b+c & a+b+c \end{vmatrix}$ [by performing $C_1 \to C_1 + C_2$ and $C_2 \to C_2 + C_3$]

$= (a+b)(b+c)\begin{vmatrix} 1 & -1 & -b \\ 1 & 1 & -a \\ -1 & 1 & a+b+c \end{vmatrix}$ [by taking $(a+b)$ and $(b+c)$ common from C_1 and C_2]

$= (a+b)(b+c)\begin{vmatrix} 0 & 0 & a+c \\ 1 & 1 & -a \\ -1 & 1 & a+b+c \end{vmatrix}$ [by performing $R_1 \to R_1 + R_3$]

$= (a+b)(b+c)[0 - 0 + (a+c)(1+1)]$ [by expanding along R_1]

$= 2(a+b)(b+c)(c+a)$

19. If $x = \cos t + \log \tan\left(\dfrac{t}{2}\right)$, $y = \sin t$, then find the values of $\dfrac{d^2y}{dt^2}$ and $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{4}$.

Sol. Given $x = \cos t + \log\left(\tan\dfrac{t}{2}\right)$

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2}$$

$$= -\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= -\sin t + \frac{1}{\sin t} = \frac{-\sin^2 t + 1}{\sin t} = \frac{\cos^2 t}{\sin t} \quad ...(i)$$

and $y = \sin t$

$$\frac{dy}{dt} = \cos t \quad ...(ii)$$

$$\Rightarrow \quad \frac{d^2y}{dt^2} = -\sin t$$

$$\left.\frac{d^2y}{dt^2}\right]_{t=\frac{\pi}{4}} = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$$

We have $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \cos t \times \dfrac{\sin t}{\cos^2 t} = \tan t$ [from (*i*) and (*ii*)]

Now $\quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(\tan t) = \sec^2 t.\dfrac{dt}{dx}$

$$= \sec^2 t.\frac{\sin t}{\cos^2 t} = \sec^4 t \cdot \sin t \quad \text{[from (i)]}$$

$$\left.\frac{d^2x}{dx^2}\right]_{t=\frac{\pi}{4}} = \sec^4\frac{\pi}{4} \cdot \sin\frac{\pi}{4} = (\sqrt{2})^4 . \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

SECTION – D

24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\dfrac{4r}{3}$. Also, find the maximum volume of cone.

Sol. Let cone of base radius x and altitude y be inscribed in a sphere of radius r.

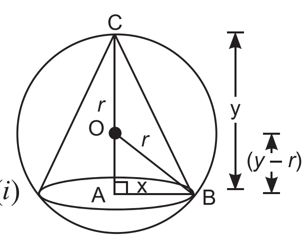
In figure, $AB = x, AC = y, OC = OB = r, OA = y - r$

In right-angled triangle OAB

$OB^2 = AB^2 + OA^2$

$\Rightarrow r^2 = x^2 + (y - r)^2$

$\Rightarrow r^2 = x^2 + y^2 - 2yr + r^2 \Rightarrow x^2 = 2yr - y^2$ ... (*i*)



Volume, $V = \dfrac{1}{3}\pi x^2 y = \dfrac{1}{3}\pi(2ry - y^2)y = \dfrac{1}{3}\pi(2ry^2 - y^3)$ [from (*i*)] ...(*ii*)

$$\frac{dV}{dy} = \frac{1}{3}\pi(4ry - 3y^2)$$

For maximum V, $\dfrac{dV}{dy} = 0$

$\Rightarrow 4ry - 3y^2 = 0 \Rightarrow 3y^2 = 4ry \Rightarrow y = \dfrac{4r}{3}$

$$\frac{d^2V}{dy^2} = \frac{1}{3}\pi\,(4\,r - 6y)$$

$$\Rightarrow \left.\frac{d^2V}{dy^2}\right]_{y=\frac{4r}{3}} = \frac{1}{3}\pi\left(4r - 6 \times \frac{4r}{3}\right) < 0$$

$\therefore$ Volume is maximum for $y = \frac{4r}{3}$ i.e. altitude is equal to $\frac{4r}{3}$.

Substituting $y = \frac{4r}{3}$ in volume, $V = \frac{1}{3}\pi(2ry^2 - y^3)$, we get

$$V_{\max} = \frac{1}{3}\pi\left[2r \cdot \frac{16r^2}{9} - \frac{64r^3}{27}\right]$$

$$= \frac{1}{3}\pi\left[\frac{96r^3 - 64r^3}{27}\right] = \frac{32}{81}\pi r^3$$

25. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1}. Hence solve the following system of equations:

$2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$

OR

Obtain the inverse of the following matrix using elementary operations:

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol. Consider $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2(0) + 3(-2) + 5(1) = -1 \neq 0$$

Hence A^{-1} exists.

Matrix formed by cofactors of each element in $|A|$

$$\begin{bmatrix} A_{11} = (-4+4) = 0, & A_{12} = (-6+4) = 2, & A_{13} = (3-2) = 1 \\ A_{21} = -(6-5) = -1, & A_{22} = (-4-5) = -9, & A_{23} = -(2+3) = -5 \\ A_{31} = (12-10) = 2, & A_{32} = -(-8-15) = 23, & A_{33} = (4+9) = 13 \end{bmatrix}$$

$$\text{Adj}\,A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = -\frac{1}{1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \qquad ...(i)$$

Consider equations,

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Corresponding matrix equation is

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

i.e. $AX = B$, Its solution is $X = A^{-1}B$

$$\Rightarrow \quad X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

or
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ $x = 1, y = 2$ and $z = 3$ is solution.

Or

Consider $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

We have $A = IA$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$ [by performing $R_1 \leftrightarrow R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}A$$ [by performing $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - 3R_1$]

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \\ 0 & -3 & 1 \end{bmatrix}A$$ [by performing $R_2 \to 2R_2 + R_3$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 & -2 \\ 2 & -1 & 1 \\ 10 & -8 & 6 \end{bmatrix}A$$ [by performing $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 + 5R_2$]

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 3 & -2 \\ 2 & -1 & 1 \\ 5 & -4 & 3 \end{bmatrix}A$$ [by performing $R_3 \to \frac{1}{2} R_3$]

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} A \quad \text{[by performing } R_1 \to R_1 + R_3 \text{ and } R_2 \to R_2 - 2R_3]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$$

SET-III (Uncommon questions to Set-I and Set-II)

SECTION – A

1. If $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, then find the matrix A.

Sol. $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow 3A = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

$\Rightarrow \quad 3A = \begin{bmatrix} 5+4 & 0+3 \\ 1+2 & 1+5 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$

$\Rightarrow \quad A = \frac{1}{3}\begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

2. Write the order and the degree of the following differential equation:

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$

Sol. Given $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$

Highest order derivative is $\frac{d^2y}{dx^2}$, and equation is written as a polynomial of derivatives.

$\therefore$ order = 2, degree = 2

3. If $f(x) = x + 1$, find $\frac{d}{dx}(fof)(x)$.

Sol. $(fof)(x) = f(f(x)) = f(x + 1) = x + 1 + 1 = x + 2$

$\frac{d}{dx}(fof)(x) = \frac{d}{dx}(x + 2) = 1.$

SECTION – B

5. Find: $\int \sin x . \log \cos x \, dx$

Sol. $\int \sin x \log(\cos x) dx = -\int \log t \, dt = -\int \log t \cdot 1 dt$ | Let $\cos x = t$ $\Rightarrow -\sin x \, dx = dt$

$= -\left[\log t . t - \int \frac{1}{t} \cdot t \, dt\right]$

$= -[t \log t - t] + C = -(\cos x)\log(\cos x) + \cos x + C$

6. Evaluate: $\int_{-\pi}^{\pi}(\mathbf{1 - x^2})\,\mathbf{\sin x \cos^2 x\, dx}$

Or

Evaluate: $\int_{-1}^{2}\frac{|x|}{x}\,dx$

Sol. Consider $\int_{-\pi}^{\pi}(1-x^2)\sin x \cos^2 x\, dx$

Let $f(x) = (1-x^2)\sin x \cos^2 x$

$f(-x) = [1-(-x^2)].\sin(-x).[\cos(-x)]^2 = (1-x^2).(-\sin x).\cos^2 x$

$= -(1-x^2)\sin x \cos^2 x = -f(x)$

$\therefore$ function is odd

$\therefore \quad \int_{-\pi}^{\pi}(1-x^2)\sin x\cos^2 x\,dx = 0 \qquad$ [As $\int_{-a}^{a} f(x)\,dx = 0$, if f is odd function]

Or

Consider $\int_{-1}^{2}\frac{|x|}{x}dx$

We have $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$= \int_{-1}^{0}\frac{-x}{x}dx + \int_{0}^{2}\frac{x}{x}\,dx = -\int_{-1}^{0}1\cdot dx + \int_{0}^{2}1\cdot dx$

$= [-x]_{-1}^{0} + [x]_{0}^{2} = -(0+1) + (2-0) = -1 + 2 = 1$

SECTION – C

13. Using properties of determinants, prove the following:

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

Sol. LHS = $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$

[by performing $C_1 \to C_1 + (C_2 + C_3)$]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$ [by taking $(a+b+c)$ common from C_1]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$ [by performing $R_3 \to R_3 - 2R_1$]

$= (a + b + c)[1\{(b - c)(a + b - 2c) - (c - a)(c + a - 2b)\}]$ [by expanding along C_1]

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= a^3 + b^3 + c^3 - 3abc = \text{RHS}$

20. Find: $\displaystyle\int \frac{\cos x}{(1 + \sin x)(2 + \sin x)}dx$

Sol. Consider $\displaystyle\int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$ | Let $\sin x = t$ $\Rightarrow \cos x\, dx = dt$

$$= \int \frac{1}{(1+t)(2+t)}dt$$

$$= \int \frac{(2+t)-(1+t)}{(1+t)(2+t)}dt$$

$$= \int\left(\frac{1}{1+t} - \frac{1}{2+t}\right)dt = \log|1 + t| - \log|2 + t| + C$$

$$= \log|1 + \sin x| - \log|2 + \sin x| + C$$

21. Solve the differential equation: $\displaystyle\frac{dy}{dx} - \frac{2x}{1 + x^2}y = x^2 + 2$

Or

Solve the differential equation: $\displaystyle(x + 1)\frac{dy}{dx} = 2e^{-y} - 1; y(0) = 0$

Sol. Consider equation $\displaystyle\frac{dy}{dx} - \frac{2x}{1+x^2}\cdot y = x^2 + 2$

Here $P(x) = \dfrac{-2x}{1+x^2}, Q(x) = x^2 + 2$

Integrating factor $= e^{\int -\frac{2x}{1+x^2}dx} = e^{-\log(1+x^2)} = e^{\log(1+x^2)^{-1}} = \dfrac{1}{1+x^2}$

Solution is (I.F.)$y = \int\{(I.F.)Q(x)\}dx$

$$\Rightarrow \quad \frac{1}{1+x^2}\cdot y = \int \frac{1}{1+x^2}\cdot(x^2+2)dx$$

$$= \int \frac{(x^2+1)+1}{1+x^2}dx = \int\left\{1 + \frac{1}{1+x^2}\right\}dx$$

$\Rightarrow \quad \dfrac{1}{1+x^2}\cdot y = x + \tan^{-1}x + C$ is the required solution.

Or

Consider equation $(x + 1)\dfrac{dy}{dx} = 2e^{-y} - 1$

$$\Rightarrow \quad \frac{dy}{2e^{-y} - 1} = \frac{dx}{x+1}$$

$$\Rightarrow \quad \int \frac{e^y dy}{2-e^y} = \int \frac{dx}{x+1}$$

Let $2 - e^y = t$

$\Rightarrow -e^y\, dy = dt$

$$\Rightarrow \quad -\int \frac{1}{t} dt = \int \frac{dx}{x+1}$$

$$\Rightarrow \quad -\log|t| = \log|x+1| + \log C$$

$$\Rightarrow \quad -\log|2-e^y| = \log|C(x+1)|$$

$$\Rightarrow \quad \log|2-e^y|^{-1} = \log|C(x+1)|$$

$$\Rightarrow \quad \frac{1}{2-e^y} = C(x+1)$$

$$\Rightarrow \quad (2-e^y)(x+1)C = 1 \qquad ...(i)$$

Given $\quad y(0) = 0$, i.e. when $x = 0, y = 0$

$$\Rightarrow \quad (2-e^0)(0+1)C = 1 \Rightarrow C = 1$$

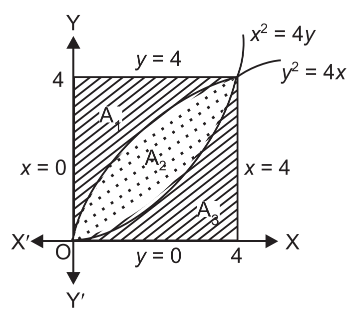
$\therefore$ From (i), $\quad (2-e^y)(x+1) = 1$ is the required solution.

26. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by sides $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts.

Or

Using integration, find the area of the triangle whose vertices are (2, 3), (3, 5) and (4, 4).

Sol. Plotting the curves $y^2 = 4x$, $x^2 = 4y$ and $x = 0$, $x = 4$, $y = 4$, $y = 0$, we notice we have three regions A_1, A_2 and A_3.



Now, A_1 : Area bounded by the curve $y^2 = 4x$, the y-axis and between $y = 0, y = 4$

$$A_1 = \int_0^4 \frac{y^2}{4} dy = \left[\frac{1}{12}y^3\right]_0^4 = \frac{64}{12} = \frac{16}{3} \text{ sq units}$$

A_2: Area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$

$$A_2 = \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx = \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4$$

$$= \left(\frac{4}{3} \times 8 - \frac{16}{3}\right) = \frac{16}{3} \text{ sq units}$$

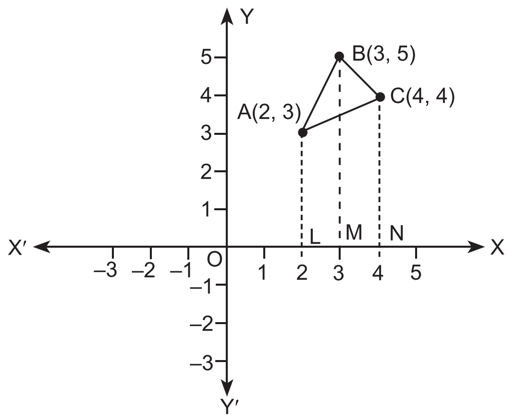
A_3 : Area bounded by the curve $x^2 = 4y$, the x-axis and between $x = 0$ and $x = 4$ is

$$A_3 = \int_0^4 \frac{x^2}{4} dx = \left[\frac{1}{12}x^3\right]_0^4 = \frac{64}{12} = \frac{16}{3} \text{ sq units}$$

$$\therefore \quad A_1 = A_2 = A_3$$

Or

Plotting the points $A(2, 3)$, $B(3, 5)$ and $C(4, 4)$, we notice we have to find ar(ABC).



$$\text{ar}(ABC) = \text{ar}(LABM) + \text{ar}(MBCN) - \text{ar}(LACN) \qquad ...(i)$$

For ar($LABM$): $A(2, 3)$, $B(3, 5)$

Equation of AB is $\quad y - 3 = \frac{5-3}{3-2}(x-2)$

$\Rightarrow \quad y - 3 = 2(x-2) \quad \Rightarrow \quad y = 2x - 1$

$\therefore \quad \text{ar}(LABM) = \int_2^3 (2x-1)\,dx = \left[x^2 - x\right]_2^3 = (9-3) - (4-2)$

$= 6 - 2 = 4 \qquad ...(ii)$

For ar($MBCN$) : $B(3, 5)$, $C(4, 4)$

Equation of BC is $\quad y - 5 = \frac{4-5}{4-3}(x-3)$

$\Rightarrow \quad y - 5 = -(x-3) \Rightarrow y = -x + 8$

$\therefore \quad \text{ar}(MBCN) = \int_3^4 (-x+8)\,dx$

$= \left[-\frac{x^2}{2} + 8x\right]_3^4 = \left(-\frac{16}{2} + 32\right) - \left(-\frac{9}{2} + 24\right)$

$= 24 + \frac{9}{2} - 24 = \frac{9}{2} \qquad ...(iii)$

For ar($LACN$); $A(2, 3)$, $C(4, 4)$

Equation of AC is $\quad y - 3 = \frac{4-3}{4-2}(x-2)$

$\Rightarrow \quad y - 3 = \frac{1}{2}(x-2) \Rightarrow y = \frac{x}{2} + 2$

$\therefore \quad \text{ar}(LACN) = \int_2^4 \left(\frac{x}{2} + 2\right) dx = \left[\frac{x^2}{4} + 2x\right]_2^4$

$= \left(\frac{16}{4} + 8\right) - \left(\frac{4}{4} + 4\right)$

$= 12 - 5 = 7$ ...(*iv*)

Substituting from (*ii*), (*iii*), (*iv*) in (*i*), we get

$$\text{ar}(ABC) = 4 + \frac{9}{2} - 7 = \frac{3}{2} \text{ sq units}$$

29. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean and variance of the number of kings.

Sol. X : number of kings,

then X can take values 0, 1, 2. Draws are without replacement.

Out of 52 cards 4 are kings and 48 other cards.

$\therefore P(0) = \frac{{}^4C_0 \times {}^{48}C_2}{{}^{52}C_2} = \frac{48 \times 47}{52 \times 51} = \frac{188}{221};$

$P(1) = \frac{{}^4C_1 \times {}^{48}C_1}{{}^{52}C_2} = \frac{4 \times 48 \times 2}{52 \times 51} = \frac{32}{221};$

$P(2) = \frac{{}^4C_2 \times {}^{48}C_0}{{}^{52}C_2} = \frac{4 \times 3}{52 \times 51} = \frac{1}{221}$

Table for probability distribution and calculation of mean and variance is

X	$P(X)$	$X.P(X)$	$X^2.P(X)$
0	$\frac{188}{221}$	0	0
1	$\frac{32}{221}$	$\frac{32}{221}$	$\frac{32}{221}$
2	$\frac{1}{221}$	$\frac{2}{221}$	$\frac{4}{221}$
Total	1	$\frac{34}{221}$	$\frac{36}{221}$

Mean $(\mu) = \sum X \,.\, P(X) = \frac{34}{221} = \frac{2}{13}$

Variance $(\sigma^2) = \Sigma X^2 P(X) - \mu^2$

$= \frac{36}{221} - \frac{4}{169}$

$= 0.1629 - 0.0237$

$= 0.1392.$